

Freshman Prize Exam

4:45 – 6:15 PM, March 11, 2009

Full proofs or explanations are expected on all answers.

Please write your netid on your exam booklet.

Problem 1) (15 points) Calculate

$$\int \frac{1}{1 + \sin x} dx.$$

Problem 2) (15 points) Let f be a three times continuously differentiable function (i.e. class C^3) of one real variable whose domain consists of all real numbers \mathbb{R} . Show that there exists $a \in \mathbb{R}$ so that

$$f(a)f'(a)f''(a)f'''(a) \geq 0.$$

Problem 3) (15 points) Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} a_n^{n/n+1}$.

Problem 4) (15 points)

- a) An unfair coin with probability p of a head is tossed repeatedly until the first head is observed. Prove that the expected number of tosses (including the toss resulting in a head) is

$$\frac{1}{p}.$$

(Don't just cite a "geometric distribution" formula, if you know one.)

- b) Suppose the coin is replaced by one with a thick edge so that there is a nonzero probability of a toss resulting in the coin landing on its edge instead of giving a head or a tail. Let p be the probability of a head and q the probability of a tail. (So $p + q < 1$.) Now toss the coin repeatedly until you have seen at least one head and at least one tail. Then stop tossing. Find a simple formula (no summations allowed) for the expected number of tosses.

Problem 5) (15 points) If every point in the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart ?

Problem 6) (15 points) Calculate

$$\int \frac{1}{x\sqrt{x^{10} + x^5 - 1}} dx.$$

Problem 7) (15 points) Let triangle ABC be an acute triangle inscribed in a circle of diameter 1. Let H be the point of intersection of the altitudes from the three vertices of the triangle. Prove that the length of side AB is $\sin C$ and the length of segment CH of the altitude is $\cos C$ where angle C is the angle between the two sides of the triangle at vertex C .

Hint: Draw a diameter of the circle through (e.g.) the vertex A .

