FRESHMAN PRIZE EXAM 2017

Full reasoning is expected. Please write your netid on your paper so we can let you know of your result. You have 90 minutes.

Problem 1. In Extracurricula all of the high school students are very into clubs. Each club contains at least three students, and there are at least two clubs. After a survey of clubs, an interesting property was noticed: for every two students there is exactly one club that both students are in, and every two clubs have exactly one student in common. What is the minimal number of students at the high school?

Problem 2. Suppose that $f''$ is continuous and that
\[
\int_0^\pi \left[ f(x) + f''(x) \right] \sin x \, dx = 2.
\]
Given that $f(\pi) = \pi$, find $f(0)$.

Problem 3. Consider all configurations of four lines $\ell_1, \ell_2, \ell_3, \ell_4$ in $\mathbb{R}^3$ such that the intersections $\ell_1 \cap \ell_2$ and $\ell_3 \cap \ell_4$ both consist of one point each, and all other pairs of lines are disjoint. What are all nonnegative integers $k$ such that for some such configuration of four lines in $\mathbb{R}^3$ there are precisely $k$ lines $\ell$ that intersect each line $\ell_i$ in exactly one point? Explain your answer.

Problem 4. Show, for $0 \leq x \leq 1$, that the error in replacing
\[
\prod_{k=0}^n 2^k \sin \left( \frac{x}{2^k} \right) = (\sin x) \left( 2 \sin \left( \frac{x}{2} \right) \right) \ldots \left( 2^n \sin \left( \frac{x}{2^n} \right) \right),
\]
by $x^{n+1}$ is at most $\frac{x^{n+3}}{3}$.

Problem 5. We say that an $(n - 1)$-tuple $a = (a_0, \ldots, a_n-2) \in (\mathbb{Z}/n)^{n-1}$ is dominated by $(0,1,\ldots,n-2)$ if there is a reordering $(a_{\pi(0)}, \ldots, a_{\pi(n-2)})$ of $a$ such that $a_{\pi(i)} \leq i$ for all $i$. (Here we think of the elements of $\mathbb{Z}/n$ as represented by $0, 1, \ldots, n - 1$ and thus $\leq$ makes sense.) Show that for any $(a_0, \ldots, a_n-2) \in (\mathbb{Z}/n)^{n-1}$ there is a $k \in \mathbb{Z}/n$ such that $(a_0 + k, \ldots, a_n-2 + k)$ is dominated by $(0, 1, \ldots, n - 2)$.

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