FRESHMAN PRIZE EXAM SOLUTIONS

Full reasoning is expected. Please write your netid on your paper so we can let you know of your result.

Problem 1. A student preparing for a math contest is doing practice problems in preparation for an exam in 10 weeks. They do at least one every day, but no more than 13 every week. Show that there exists a sequence of days where the student tried exactly 9 problems.

Solution: Let $a_i$ be the number of problems practiced on days 1 to $i$. Then we have

$$a_1 < a_2 < \cdots < a_{70} \leq 10 \times 13 = 130,$$

where the last inequality is because over the 10 weeks, they do no more than 13*10 problems. We have $a_1 + 9 < a_2 + 9 < \cdots < a_{70} + 9 \leq 139$. Combining these two sequences we have 140 numbers (70+70) less than 140 and greater than 0. Thus two must be equal, more over they must be $a_i = a_j + 9$, so that $a_i - a_j = 9$, i.e. from day $j+1$ to day $i$, a total of 9 problems where done.

Problem 2. Given a circle with $n$ points on the boundary, join all pairs of points by chords of the circle. Suppose that no more than two chords intersect at any given point inside the circle. How many intersection points are there?

Solution: Given 4 distinct points, say A,B,C,D, arranged counterclockwise around the boundary, then the chord between two diagonally opposite points, say A and C, separates the circle and its inside into two regions, one of which contains B and the other of which contains D. Consequently the chord BD must meet the chord AC in an intersection point.

What this implies is that there is a bijection between the sets of two intersecting chords and the sets of ordered lists of their endpoints. Thus, the number of intersections is $n\choose 4$.

Problem 3. Suppose that $(x_n)$ is a sequence satisfying $\lim_{n \to \infty} (x_n - x_{n-1}) = 0$, then show that

$$\lim_{n \to \infty} \frac{x_n}{n} = 0.$$

Solution: Given $\epsilon > 0$, there exists $N$ such that $|x_n - x_{n-1}| < \frac{1}{2}\epsilon$ for all $n > N$. Thus $|x_n| \leq |x_N| + \frac{1}{2}\epsilon(n - N)$ and

$$\frac{|x_n|}{n} \leq \frac{|x_N| - \frac{1}{2}\epsilon N}{n} + \frac{1}{2}\epsilon.$$

When $2\frac{|x_N| - \frac{1}{2}\epsilon N}{\epsilon} < n$, we have $\frac{|x_N| - \frac{1}{2}\epsilon N}{n} < \frac{1}{2}\epsilon$. Set $N' = \max\{N, 2\frac{|x_N| - \frac{1}{2}\epsilon N}{\epsilon}\}$.

When $n > N'$, we have $\frac{|x_n|}{n} < \epsilon$.

Problem 4. Let $f$ and $g$ be continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Assume

1. there is no $x \in \mathbb{R}$ satisfying $f(x) = g(x)$,
2. for all $x \in \mathbb{R}$ we have $f(g(x)) = g(f(x))$.

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Prove that there is no \( x \in \mathbb{R} \) satisfying \( f(f(x)) = g(g(x)) \)

**Solution:** Define

\[ h(x) = f(x) - g(x) . \]

If \( h(a) > 0 \) and \( h(b) < 0 \) for some \( a, b \in \mathbb{R} \), then \( h(c) = 0 \) for some value \( c \in \mathbb{R} \) (Bolzano’s Theorem or Intermediate value theorem). So \( h \) is always positive or always negative.

- If \( f(x) > g(x) \) for all \( x \in \mathbb{R} \), then for any \( x \in \mathbb{R} \) we have
  \[ f(f(x)) > g(f(x)) = f(g(x)) . \]
- If \( f(x) < g(x) \) for all \( x \in \mathbb{R} \), then for any \( x \in \mathbb{R} \) we have
  \[ f(f(x)) < g(f(x)) = f(g(x)) . \]

**Problem 5.** Show that

\[ \sum_{n=1}^{\infty} \cos^{-1} \left( \frac{1 + \sqrt{n^2 + 2n\sqrt{(n+1)^2 + 2(n+1)}}}{(n+1)(n+2)} \right) = \frac{\pi}{6} . \]

**HINT:** \( \cos(A - B) = \cos A \cos B + \sin A \sin B \).

**Solution:** Let

\[ A_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{\sqrt{n^2 + 2n\sqrt{(n+1)^2 + 2(n+1)}}}{n+1} . \]

Define \( \{ \alpha_n \} \) (each \( \alpha_n \) between 0 and \( \frac{\pi}{2} \)) so that

\[ \cos(\alpha_n) = \frac{1}{n+1} , \]

\[ \sin(\alpha_n) = \frac{\sqrt{n^2 + 2n}}{n+1} . \]

Then

\[ A_n = \cos(\alpha_n) \cos(\alpha_{n+1}) + \sin(\alpha_n) \sin(\alpha_{n+1}) = \cos(\alpha_{n+1} - \alpha_n) . \]

Since \( \cos x \) is a decreasing function for \( 0 \leq x \leq \frac{\pi}{2} \), note that \( \alpha_n < \alpha_{n+1} \) for each \( n \).

We have

\[ \sum_{n=1}^{N} \cos^{-1}(A_n) = \sum_{n=1}^{N} \cos^{-1}(\cos(\alpha_{n+1} - \alpha_n)) \]

\[ = \sum_{n=1}^{N} (\alpha_{n+1} - \alpha_n) \]

\[ = \alpha_{N+1} - \alpha_1 \]

\[ = \cos^{-1}(\frac{1}{2+N}) - \cos^{-1}(\frac{1}{2}) . \]

So, letting \( N \to \infty \), we obtain

\[ \sum_{n=1}^{\infty} \cos^{-1}(A_n) = \cos^{-1}(0) - \cos^{-1}(\frac{1}{2}) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} . \]

**Problem 6.** Prove that for \( n \geq 2 \), the real number

\[ H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \]

is not an integer.

**HINT:** If you write \( H_n \) as a fraction \( \frac{p}{q} \), you may want to think about the largest
power of 2 dividing the numerator and the largest power of 2 dividing the denominator.

Solution: By combining fractions with common denominator $n!$, we have

$$H_n = \frac{p_n}{q_n},$$

where

$$p_n = \frac{n!}{1} + \frac{n!}{2} + \frac{n!}{3} + \frac{n!}{4} + \ldots + \frac{n!}{n}$$

$$q_n = n!$$

Let $\frac{p}{q}$ be the fraction $\frac{p_n}{q_n}$ reduced to lowest terms. We will show that the denominator $q$ is even and the numerator $p$ is odd, thereby establishing that $H_n$ is not a whole number.

Let $2^e$ be the largest power of 2 dividing $n!$. Then each of the numerator terms

$$\frac{n!}{k}$$

for $1 \leq k \leq n$ will be divisible by a power of 2 which is less than or equal to $2^e$.

Let $f$ denote the power 2 must be raised to so that $2^f \leq n < 2^{f+1}$. Obviously $e \geq f \geq 1$ for $n \geq 2$. In fact every term $\frac{n!}{2^f}$ for $k \neq 2^f$ (still $k \leq n$) is divisible by $2^{e-f+1}$ while $\frac{n!}{2^f}$ is divisible by $2^{e-f}$ but no higher power of 2. (This is because there is only one nonzero multiple of $2^f$ smaller than $2^{f+1}$.)

Consequently the numerator $p_n$ before fraction reduction is of the form

$$2^{e-f+1} \text{(integer)} + 2^{e-f} \text{(odd integer)} = 2^{e-f} \text{(a different odd integer)}.$$

Since the denominator $q_n$ is of the form $2^e \text{(odd integer)}$, it is clear that the reduced form $\frac{p}{q}$ is of the form $\frac{(\text{odd integer})}{2^f \text{(odd integer)}}$.

In particular $p$ is odd while $q$ is even showing the sum is not a whole number.

Please Note: The above argument is in part based on some of the discussion at:

- http://math.stackexchange.com/questions/2746/is-there-an-elementary-proof-that-sum-limits-k-1n-frack-is-never-an-int
- http://math.uga.edu/~pete/4400intro.pdf (page 13)
- http://www.math.uconn.edu/~kconrad/blurbs/ugradnumthy/padicharmonicsum.pdf (Theorem 1)