FRESHMAN PRIZE EXAM

Full reasoning is expected. Please write your netid on your paper so we can let you know of your result.

Problem 1. **A Sum** Show that

\[ \sum_{n=1}^{\infty} \frac{1}{n^3} = \int_0^1 \left( \int_0^y \left( \int_0^{(1-x)y} dx \right) dy \right) dz. \]

(Here you need not prove convergence rigorously.)

**Solution Sketch:** Start with

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \]

and work with the integrals successively.

\[ \int_0^1 \left( \int_0^y \left( \int_0^{(1-x)y} dx \right) dy \right) dz = \sum_{n=0}^{\infty} \int_0^1 \left( \int_0^y \left( \int_0^{x^ny} dx \right) dy \right) dz \ldots \]

Problem 2. **Binomial Coefficient Sums** It is true (and often studied in probability) that

\[ \sum_{k=0}^{n} \binom{n}{k} 2^{-n} = 1, \]

\[ \sum_{k=0}^{n} k \binom{n}{k} 2^{-n} = \frac{n}{2}, \]

\[ \sum_{k=0}^{n} \left( k - \frac{n}{2} \right) \binom{n}{k} 2^{-n} = \frac{n}{4}. \]

where \( \binom{n}{k} \) is the binomial coefficient counting the number of combinations of \( n \) things taken \( k \) at a time. Find closed form formulae for

\[ \sum_{k=0}^{n} \left( k - \frac{n}{2} \right)^3 \binom{n}{k} 2^{-n} \quad \text{and} \quad \sum_{k=0}^{n} \left( k - \frac{n}{2} \right)^4 \binom{n}{k} 2^{-n}. \]

**Solution Sketch:** An inductive approach based on the binomial coefficient identity

\[ k \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = \frac{n}{k} \binom{n-1}{k-1} = n \binom{n-1}{k-1} \]

is readily workable.

Even without that, an oddness symmetry argument based on

\[ \sum_{k=0}^{n} \left( k - \frac{n}{2} \right)^3 \binom{n}{k} 2^{-n} = \sum_{k<\frac{n}{2}} \left( k - \frac{n}{2} \right)^3 \binom{n}{k} 2^{-n} + \sum_{k>\frac{n}{2}} \left( k - \frac{n}{2} \right)^3 \binom{n}{k} 2^{-n} \]

shows that this sum is 0. (One could formally substitute \( k = n - j \) in one of the summations on the right if one wanted to show this more formally.)

Date: March 6, 2013.
To tackle the fourth power sum, note
\[\sum_{k=0}^{n} (k - \frac{n}{2})^4 \left( \begin{array}{c} n \\ k \end{array} \right) 2^{-n} = \sum_{k=0}^{n} k^4 \left( \begin{array}{c} n \\ k \end{array} \right) 2^{-n} + \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{n}{2} k^3 2^{-n} \]
\[= \sum_{k=0}^{n} (k - \frac{n}{2})^3 \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right) 2^{-n} - \frac{n}{2} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) k^3 2^{-n} \]
\[= n \sum_{j=0}^{n-1} \left( j + 1 - \frac{n}{2} \right)^3 \left( \begin{array}{c} n-1 \\ j \end{array} \right) 2^{-n} - 0 \]
the last step being based on the substitution \( j = k - 1 \). Continuing, basing our reasoning on the corresponding degree at most 3 sums for \( n - 1 \),
\[\sum_{k=0}^{n} (k - \frac{n}{2})^4 \left( \begin{array}{c} n \\ k \end{array} \right) 2^{-n} = \frac{n}{2} \sum_{j=0}^{n-1} \left( j + 1 - \frac{n}{2} \right)^3 \left( \begin{array}{c} n-1 \\ j \end{array} \right) 2^{-(n-1)} \]
\[= \frac{n}{2} \sum_{j=0}^{n-1} \left( j - \frac{n-1}{2} \right)^3 \left( \begin{array}{c} n-1 \\ j \end{array} \right) 2^{-(n-1)} + \frac{3n}{4} \sum_{j=0}^{n-1} \left( j - \frac{n-1}{2} \right)^2 \left( \begin{array}{c} n-1 \\ j \end{array} \right) 2^{-(n-1)} + \frac{3n}{8} \sum_{j=0}^{n-1} \left( j - \frac{n-1}{2} \right) \left( \begin{array}{c} n-1 \\ j \end{array} \right) 2^{-(n-1)} + \frac{n}{16} \sum_{j=0}^{n-1} \left( \begin{array}{c} n-1 \\ j \end{array} \right) 2^{-(n-1)} \]
\[= 0 + \frac{3n}{4} \cdot \frac{(n-1)}{4} + \frac{3n}{8} \cdot 0 + \frac{n}{16} \cdot 1 \]
\[= \frac{n(3n-2)}{16} \]

**Remark:** A generating function approach based on differentiating
\[\left( \frac{t}{2} + 1 \right)^n = \sum_{k=0}^{n} t^k \left( \begin{array}{c} n \\ k \end{array} \right) 2^{-n} \]
several times and plugging in \( t = 1 \) is also possible.

**Problem 3. A Fixed Point Condition** Let \( f : [0,1] \to R \) be a continuous function satisfying
\[2 \int_{0}^{1} f(x) \, dx = 1. \]
Show that there is an \( x_0 \) in the open interval \((0,1)\) for which \( f(x_0) = x_0 \).

**Solution Sketch:** Consider the function \( g : R \to R \) defined by
\[g(x) = f(x) - x. \]
By hypothesis
\[\int_{0}^{1} g(x) \, dx = 0. \]
If \( g(x) \) is identically 0 every point in \((0,1)\) is the required fixed point. If \( g(x) \) is not identically 0, then it must be positive somewhere and negative somewhere else since its average value is 0. By continuity, then there exist \( a, b \in (0,1) \) with \( g(a) \) and \( g(b) \) of opposite sign. Now the intermediate value theorem gives us a point \( x_0 \) in \((0,1)\) where \( g(x_0) = 0 \) and \( f(x_0) = x_0 \).

**Remark:** One could also base a briefer version of the argument on a mean value theorem for integrals.
Problem 4.  

An Inequality 

Let $p(x)$ be a cubic polynomial, 

$$p(x) = a_0x^3 + a_1x^2 + a_2x + a_3$$

with all coefficients $a_0$, $a_1$, $a_2$, and $a_3$ real and positive.

Prove that for any positive numbers $a$ and $b$, 

$$P(\sqrt{ab}) \leq \sqrt{P(a)P(b)}.$$ 

Hint: Remember the Cauchy-Schwartz inequality which says for vectors that 

$$\vec{v} \cdot \vec{w} \leq ||\vec{v}|| ||\vec{w}||.$$ 

Solution Sketch: This works for a polynomial of any degree. Conceptually the inner product 

$$(x_0, x_1, ..., x_n) \cdot (y_0, y_1, ..., y_n) = a_0x_0y_0 + a_1x_1y_1 + ... + a_nxNy_n$$

would be best with vectors 

$$(a^{n/2}, a^{(n-1)/2}, ..., 1) \text{ and } (b^{n/2}, b^{(n-1)/2}, ..., 1)$$

but one can easily enough throw in factors of $\sqrt{a_i}$ and confine oneself to the usual $R^{n+1}$ dot product. (The dot product becomes $P(\sqrt{ab})$ and the lengths squared become $P(a)$ and $P(b)$.)

Problem 5.  

Iteration of Quadratic Maps 

Let $\mu$ be a real number satisfying $-\frac{3}{4} < \mu < \frac{1}{4}$. Consider the quadratic polynomial $f(x) = x^2 + \mu$ and consider for any value $x_0$ the sequence 

$$x_0, x_1, x_2, ..., x_n, ...$$

where $x_1 = f(x_0)$, $x_2 = f(x_1)$, and in general $x_{i+1} = f(x_i)$ for $i \geq 0$.

Show for any $\mu$ in the range given above, there is a closed interval $[a, b]$ of positive length so that whenever $x_0 \in [a, b]$, both the following hold:

(1) Every number $x_n$ in the sequence $\{x_n\}_{n=0}^{\infty}$ also lies in the interval $[a, b]$.

(2) The limit $\lim_{n \to \infty} x_n$ exists.

(The interval $[a, b]$ may depend on $\mu$.)

Solution Sketch: If the limit exists, by continuity it must be a fixed point of $f(x)$. The quadratic formula applied to $x^2 + \mu = x$ shows under the stated range of $\mu$ there are two distinct real roots $x_\pm$ and one of them is strictly between $-\frac{1}{2}$ and $\frac{1}{2}$. (The other is greater than 1.)

Since $f'(x) = 2x$, the root $x_-$ strictly between $-\frac{1}{2}$ and $\frac{1}{2}$ satisfies $|f'(x_-)| < 1$, and the mean value estimate $|f(x) - f(y)| \leq R|x - y|$ for $x, y$ in a symmetric interval about $x_-$ on which $|f'(x)| \leq R < 1$ gives what we need. In particular, since $f(x_-) = x_-$, 

$$|f(x) - x_-| \leq R|x - x_-|$$

shows such an interval is invariant under the iteration, and that $\lim_{n \to \infty} x_n = x_-$. 

Problem 6.  

A Functional Equation 

Consider a continuous function $f : R \to R$ satisfying $(f \circ f \circ f)(x) = x$ for all real numbers $x$. Prove that $f(x) = x$. 

Solution Sketch: The function $f(x)$ must be bijective since a failure of one-to-one-ness or onto-ness would imply the same failure for $(f \circ f \circ f)$. Or from a more advanced point of view, because $f$ has a left and right inverse $(f \circ f)$.

So $f$ has to be monotone. Since the composition of decreasing functions is increasing (and decreasing $\circ$ increasing = decreasing again), we see that $f$ must be increasing.

Fix some $x \in R$. We want to show $f(x) = x$. If not there are two possibilities:

$x < f(x)$:
Set $x_1 = f(x)$ and $x_2 = (f \circ f)(x) = f(x_1)$. Then

$x < f(x) \Rightarrow f(x) < (f \circ f)(x) \Leftrightarrow x_1 < x_2 \Rightarrow f(x_1) < f(x_2) = x \Leftrightarrow x_2 < x$.

We thus have the impossible chain $x < x_1 < x_2 < x$ and so this case cannot happen.

$x > f(x)$:
Again set $x_1 = f(x)$ and $x_2 = (f \circ f)(x) = f(x_1)$ and now reverse the inequalities above. Specifically

$x > f(x) \Rightarrow f(x) > (f \circ f)(x) \Leftrightarrow x_1 > x_2 \Rightarrow f(x_1) > f(x_2) = x \Leftrightarrow x_2 > x$.

We thus have the impossible chain $x > x_1 > x_2 > x$ and so this case also cannot happen.

Since $x$ was arbitrary, this proves $f(x) = x$ for all $x$. 