

## Freshman Prize Exam Solutions

5:45 – 7:15 PM, April 9, 2008

Full proofs or explanations are expected on all answers.

*Please write your netid on your exam booklet.*

**Problem 1) (10 points)** Let  $a$  be a natural number, and  $b = 10^a - 4$ . Prove that  $10^b - 1$  is divisible by 7.

**Solution:** We are interested in  $3^b - 1 \pmod{7}$ . Since  $3^6$  is  $1 \pmod{7}$ , (by Fermat's theorem or direct calculation), it would be enough to show  $b$  is divisible by 6. This is equivalent to  $b$  being divisible by 2 and 3. Reducing  $b = 10^a - 4 \pmod{2}$  and  $\pmod{3}$  immediately give the required divisibility properties of  $b$ .

**Problem 2) (15 points)**

a) Show that

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx.$$

b) Prove that if  $n$  is a multiple of 4, then there are rational numbers  $\alpha$  and  $\beta$  so that

$$\int_0^1 \frac{x^n(1-x)^n}{1+x^2} dx = \alpha + \beta\pi.$$

c) Find the value of  $\beta$  for  $n$  such a multiple of 4. ( $\beta$  depends on  $n$ .)

d) Show that

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{x^n(1-x)^n}{1+x^2} dx = 0.$$

**Solution:**

a) Long division gives

$$x^4(1-x)^4 = (1+x^2)P_1(x) - 4$$

where  $P_1(x) = x^6 - 4x^5 + 5x^4 - 4x^2 + 4$ . Now

$$\int_0^1 P_1(x) dx = \frac{1}{7} - \frac{4}{6} + 1 - \frac{4}{3} + 4 = \frac{22}{7}.$$

and

$$\int_0^1 \frac{1}{1+x^2} = \arctan(x)|_0^1 = \frac{\pi}{4}$$

giving the desired result.

- b) We prove by induction on  $m$  that there is a polynomial  $P_m(x)$  with rational coefficients and an integer  $c_m$  so that

$$x^{4m}(1-x)^{4m} = (1+x^2)P_m(x) + c_m.$$

The case  $m = 1$  was proven in part a). Suppose the above is true for  $m$ . Then

$$x^{4m+4}(1-x)^{4m+4} = (1+x^2)P_m(x)(x^4(1-x)^4) + c_mx^4(1-x)^4.$$

Using  $x^4(1-x)^4 = (1+x^2)P_1(x) - 4$ , we have

$$x^{4m+4}(1-x)^{4m+4} = (1+x^2)(x^4(1-x)^4P_m(x) + c_mP_1(x)) - 4c_m$$

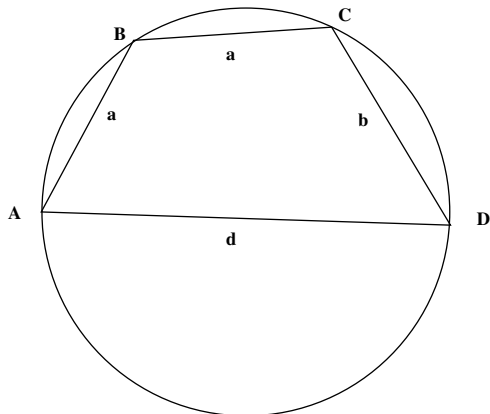
as was to be proven. (Since  $\int_0^1 P_{m+1}(x) dx$  is rational and  $\int_0^1 \frac{1}{1+x^2} = \frac{\pi}{4}$ , we have  $\alpha$  and  $\beta$  rational as well.)

- c) The above showed  $c_{m+1} = -4c_m$ , so together with  $c_1 = -4$ , we clearly have  $\beta = c_m = (-4)^m$  where  $n = 4m$ .
- d) The maximum of  $x(1-x)$  is  $\frac{1}{4}$  when  $x = \frac{1}{2}$  (and  $1+x^2 \geq 1$ ), so the integral is positive and clearly less than  $2^{-2n}$ . Therefore it approaches 0 as  $n \rightarrow \infty$ .

**Problem 3) (15 points)** Suppose ABCD is a cyclic quadrilateral, as shown, with side AD =  $d$ , where  $d$  is the diameter of the circle. AB =  $a$ , BC =  $a$ , and CD =  $b$ . Suppose  $a$ ,  $b$ , and  $d$  are integers with  $a \neq b$ .

- a) Prove that  $d$  cannot be a prime number.
- b) Determine the minimum value of  $d$ .

*Hint: Express the angle ADC in terms of the angle ABC to get a simple equation involving  $a$ ,  $b$ , and  $d$ .*



**Solution:** (This was question 10 on the 1999 Euclid Examination in Canada. A more detailed solution is available at the Euclid website in Waterloo.)

- a) We can use both the cosine law for triangle ABC and the Pythagorean Theorem for triangle ACD to express the square of diagonal AC. When we do this and simplify, we obtain:

$$2a^2 = d(d - b).$$

If  $d$  were an odd prime  $p$ , then  $a^2$  would be divisible by  $p$ , implying that  $a$  is divisible by  $p$ . So the left hand side would be divisible by  $p^2$  while the right hand side was not, a contradiction.

The case  $d = 2$  is also impossible since  $2 = a^2 + b$  is impossible in positive integers with  $a \neq b$  (as was specified.)

- b) Ruling out the primes for  $d$  leaves  $d = 1, 4, 6, 8, \dots$ . The cases  $d = 1, 4$ , or  $6$  are easily seen to not work in integers. ( $a \neq b$  is sometimes needed.) The next possibility,  $d = 8$  leads to the realizable  $b = 7$  and  $a = 2$ . So  $d = 8$  is the minimum  $d$ .

**Problem 4) (10 points)** Show that every real number  $x$  satisfies

$$-1 < \sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1} < 1.$$

**Solution:** First consider the case  $x > 0$ . Note that

$$0 < \sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1} < 1$$

would immediately follow upon division by  $\sqrt{x^2 + x + 1} + \sqrt{x^2 - x + 1}$  if we could establish

$$2x < \sqrt{x^2 + x + 1} + \sqrt{x^2 - x + 1}.$$

But this latter statement is clearly true since

$$\begin{aligned} 2x &= \left(x + \frac{1}{2}\right) + \left(x - \frac{1}{2}\right) \\ &\leq \sqrt{x^2 + x + \frac{1}{4}} + \sqrt{x^2 - x + \frac{1}{4}} \\ &< \sqrt{x^2 + x + 1} + \sqrt{x^2 - x + 1}. \end{aligned}$$

Replacing  $x$  by  $-x$  shows us

$$1 > \sqrt{x^2 - x + 1} - \sqrt{x^2 + x + 1} > 0$$

for  $x < 0$ .

Combining the two cases with the value when  $x = 0$  gives the stated inequality.

**Problem 5) (10 points)** Let  $f(x)$  be a continuous function satisfying

$$\begin{aligned}\int_0^1 f(x) dx &= 0 \\ \int_0^1 xf(x) dx &= 0 \\ \int_0^1 x^2 f(x) dx &= 1.\end{aligned}$$

Prove that the maximum value of  $f(x)$  on the interval  $[0, 1]$  is at least 12.

**Solution:** Let  $M$  be the maximum value of  $f$  on the interval (which exists since  $f$  is continuous and the interval is compact.) The hypotheses imply that

$$\int_0^1 \left(x - \frac{1}{2}\right)^2 f(x) dx = 1.$$

But

$$\int_0^1 \left(x - \frac{1}{2}\right)^2 f(x) dx \leq M \int_0^1 \left(x - \frac{1}{2}\right)^2 dx \leq \frac{M}{12}.$$

So  $\frac{M}{12} \geq 1$  and  $M \geq 12$ .

**Problem 6) (15 points)** Let  $P(x)$  be a polynomial of degree  $n$  such that  $P(t) = 2^t$  for  $t = 1, 2, \dots, n + 1$ . Is the value of  $P(n + 2)$  determined? If yes, compute it.

**Solution:**

By the fundamental theorem of algebra, two polynomials of degree  $n$  which agree at  $n + 1$  distinct points are identical. So  $P(n + 2)$  is determined.

Let  $P_n$  be the polynomial of degree  $n$  specified by the problem. So  $P_n(x) = 2^x$  for  $1 \leq x \leq n + 1$ .

Define  $Q_{n-1}(x) = P_n(x + 1) - P_n(x)$ .  $Q_{n-1}(x)$  is of degree  $n - 1$  and for  $1 \leq x \leq n$ ,  $Q_{n-1}(x) = 2^{x+1} - 2^x = 2^x$ . So  $Q_{n-1} = P_{n-1}$ . We refer to

$$P_{n-1}(x) = P_n(x + 1) - P_n(x)$$

as the basic recursive identity satisfied by  $P_n$ .

We now claim  $P_n(n + 2) = 2^{n+2} - 2$  and prove this inductively:

**Case  $n = 1$ :**  $P_1(x) = 2x$  by inspection (since we already have uniqueness) and so  $P_1(3) = 6 = 2^{1+2} - 2$  as claimed.

**Suppose true for  $n = k - 1$ :** Then by the basic recursive identity, defining property of  $P_n$ , and inductive hypothesis, we have

$$P_k(k + 2) - P_k(k + 1) = P_{k-1}(k + 1)$$

or

$$\begin{aligned}P_k(k+2) &= P_k(k+1) + P_{k-1}(k+1) \\ &= 2^{k+1} + 2^{k+1} - 2 \\ &= 2^{k+2} - 2\end{aligned}$$

proving the case  $n = k$  of the induction.

**Remark:** Two other interesting solutions were constructed by students taking the exam. One (due to Michael Shu) was based on writing  $P_m(x)$  as  $2 + (x-1)R_1(x) + (x-1)(x-2)R_2(x) \dots$  and thinking about the conditions  $P_m(1) = 2$ ,  $P_m(2) = 4$ ,  $\dots$  sequentially. The other (by John Sheridan) was based on observing that if  $g(x) = (x-1)(x-2)\dots(x-n)$ , then the functions

$$a_i(x) = \frac{g(x)}{(x-i)g'(x)}$$

for  $i = 1, 2, \dots, n$  satisfy  $a_i(j) = \delta_i^j$  for  $1 \leq j \leq n$  and so give a nice way to do Lagrangian interpolation.

**Problem 7) (15 points)** Let  $a, b \in \mathbb{C}$  with  $|a| > |b|$ . Show that the locus in  $\mathbb{C}$  of the equation  $|az + b\bar{z}| = 1$  is an ellipse, and find the ratio of the major axis to the minor axis. (*Here  $\mathbb{C}$  denotes the complex numbers.*)

**Solution:** If  $a = a_0 e^{i\phi}$  with  $a_0$  real and positive, set  $c = b e^{-i\phi}$  and note we are equivalently looking at

$$|a_0 z + c\bar{z}| = 1.$$

Letting  $c = c_0 + ic_1$  (with all  $c_i$  real) and using  $|w|^2 = w\bar{w}$ , we have

$$\begin{aligned}|a_0 z + c\bar{z}|^2 &= 1 \\ (a_0 z + c\bar{z})(a_0 \bar{z} + \bar{c}z) &= 1 \\ (a_0^2 + |c|^2)|z|^2 + 2a_0 \operatorname{Real}(z^2 \bar{c}) &= 1\end{aligned}$$

Writing  $z = x + iy$  with  $x, y \in \mathbb{R}$ , we have

$$\begin{aligned}|z|^2 &= x^2 + y^2 \\ z^2 &= x^2 - y^2 + i(2xy) \\ \operatorname{Real}(z^2 \bar{c}) &= c_0(x^2 - y^2) - c_1(2xy)\end{aligned}$$

So our equation becomes

$$1 = (a_0^2 + |c|^2)(x^2 + y^2) + 2a_0 c_0(x^2 - y^2) - 4a_0 c_1 xy.$$

The right hand side is the quadratic form associated to the symmetric matrix

$$A = \begin{pmatrix} (a_0^2 + |c|^2) + 2a_0c_0 & -2a_0c_1 \\ -2a_0c_1 & (a_0^2 + |c|^2) - 2a_0c_0 \end{pmatrix}$$

The matrix  $A$  has determinant

$$\begin{aligned} (a_0^2 + |c|^2 + 2a_0c_0)(a_0^2 + |c|^2 - 2a_0c_0) - 4a_0^2c_1^2 &= (a_0^2 + |c|^2)^2 - 4a_0^2|c|^2 \\ &= (a_0^2 + |c|^2 + 2a_0|c|)(a_0^2 + |c|^2 - 2a_0|c|) \\ &= (a_0 + |c|)^2(a_0 - |c|)^2 \end{aligned}$$

and trace  $2(a_0^2 + |c|^2)$ . So by inspection (or the quadratic formula), the eigenvalues of  $A$  are  $(a_0 + |c|)^2$  and  $(a_0 - |c|)^2$ . Hence there exists a rotation of coordinates from  $(x, y)$  to new coordinates  $(\tilde{x}, \tilde{y})$  so that in the new coordinates, our locus is

$$(a_0 + |c|)^2\tilde{x}^2 + (a_0 - |c|)^2\tilde{y}^2 = 1$$

which is obviously an ellipse with semimajor axis  $(a_0 - |c|)^{-1}$  and semiminor axis  $(a_0 + |c|)^{-1}$ . In terms of the original data, these are  $(|a| \pm |b|)^{-1}$  and the ratio of major axis to minor axis is

$$\frac{|a| + |b|}{|a| - |b|}.$$

**Remark:** This is useful in the theory of quasiconformal mappings; see e.g. page 113 of *Teichmüller Theory Volume 1* by John Hubbard.