Problem 1) (10 points) Let $a$ be a natural number, and $b = 10^a - 4$. Prove that $10^b - 1$ is divisible by 7.

Problem 2) (15 points)

a) Show that
\[
\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx.
\]

b) Prove that if $n$ is a multiple of 4, then there are rational numbers $\alpha$ and $\beta$ so that
\[
\int_0^1 \frac{x^n(1-x)^n}{1+x^2} \, dx = \alpha + \beta \pi.
\]

c) Find the value of $\beta$ for $n$ such a multiple of 4. ($\beta$ depends on $n$.)

d) Show that
\[
\lim_{n \to \infty} \int_0^1 \frac{x^n(1-x)^n}{1+x^2} \, dx = 0.
\]

Problem 3) (15 points) Suppose ABCD is a cyclic quadrilateral, as shown, with side $AD = d$, where $d$ is the diameter of the circle. $AB = a$, $BC = a$, and $CD = b$. Suppose $a$, $b$, and $d$ are integers with $a \neq b$.

a) Prove that $d$ cannot be a prime number.

b) Determine the minimum value of $d$.

*Hint: Express the angle $ADC$ in terms of the angle $ABC$ to get a simple equation involving $a$, $b$, and $d$. 
Problem 4) (10 points) Show that every real number \( x \) satisfies

\[-1 < \sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1} < 1.\]

Problem 5) (10 points) Let \( f(x) \) be a continuous function satisfying

\[
\int_0^1 f(x) \, dx = 0
\]

\[
\int_0^1 xf(x) \, dx = 0
\]

\[
\int_0^1 x^2f(x) \, dx = 1.
\]

Prove that the maximum value of \( f(x) \) on the interval \([0, 1]\) is at least 12.

Problem 6) (15 points) Let \( P(x) \) be a polynomial of degree \( n \) such that \( P(t) = 2^t \) for \( t = 1, 2, ..., n+1 \). Is the value of \( P(n+2) \) determined? If yes, compute it.

Problem 7) (15 points) Let \( a, b \in \mathbb{C} \) with \(|a| > |b|\). Show that the locus in \( \mathbb{C} \) of the equation \(|az + bz| = 1\) is an ellipse, and find the ratio of the major axis to the minor axis. (Here \( \mathbb{C} \) denotes the complex numbers.)