Freshman Prize Exam

5:45 – 7:15 PM, April 9, 2008

Full proofs or explanations are expected on all answers.
Please write your netid on your exam booklet.

Problem 1) (10 points) Let a be a natural number, and $b = 10^a - 4$. Prove that $10^b - 1$ is divisible by 7.

Problem 2) (15 points)

a) Show that

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx.$$

b) Prove that if n is a multiple of 4, then there are rational numbers α and β so that

$$\int_0^1 \frac{x^n(1-x)^n}{1+x^2}\, dx = \alpha + \beta\pi.$$

c) Find the value of β for n such a multiple of 4. (β depends on n.)

d) Show that

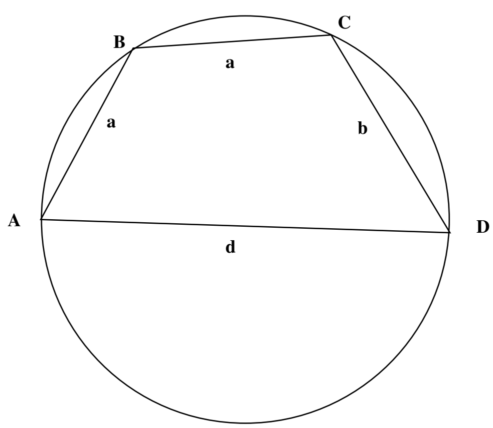
$$\lim_{n\to\infty} \int_0^1 \frac{x^n(1-x)^n}{1+x^2}\, dx = 0.$$

Problem 3) (15 points) Suppose ABCD is a cyclic quadrilateral, as shown, with side AD $= d$, where d is the diameter of the circle. AB $= a$, BC $= a$, and CD $= b$. Suppose a, b, and d are integers with $a \neq b$.

a) Prove that d cannot be a prime number.

b) Determine the minimum value of d.

Hint: Express the angle ADC in terms of the angle ABC to get a simple equation involving a, b, and d.



Problem 4) (10 points) Show that every real number x satisfies

$$-1 < \sqrt{x^2+x+1} - \sqrt{x^2-x+1} < 1.$$

Problem 5) (10 points) Let $f(x)$ be a continuous function satisfying

$$\int_0^1 f(x)\ dx = 0$$
$$\int_0^1 xf(x)\ dx = 0$$
$$\int_0^1 x^2 f(x)\ dx = 1.$$

Prove that the maximum value of $f(x)$ on the interval $[0,1]$ is at least 12.

Problem 6) (15 points) Let $P(x)$ be a polynomial of degree n such that $P(t) = 2^t$ for $t = 1, 2, ..., n+1$. Is the value of $P(n+2)$ determined? If yes, compute it.

Problem 7) (15 points) Let $a, b \in \mathbb{C}$ with $|a| > |b|$. Show that the locus in $\mathbb{C}$ of the equation $|az + b\overline{z}| = 1$ is an ellipse, and find the ratio of the major axis to the minor axis. *(Here $\mathbb{C}$ denotes the complex numbers.)*