Freshmen Prize Exam 2007

1. Calculate
$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{x(2-x)}+1}\,dx.$$
Answer. First complete the square giving
$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{x(2-x)}+1}\,dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{1-(x-1)^2}+1}\,dx.$$
Now let $u = x - 1$ obtaining
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}+1}\,du.$$
The substitution $u = \sin v$ takes us to
$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos v}{\cos v + 1}\,dv = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1 - \frac{1}{\cos v + 1}\,dv.$$
The double angle formula $\cos v = 2\cos^2 \frac{v}{2} - 1$ takes us to
$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 1 - \frac{1}{2\cos^2 \frac{v}{2}}\,dv = v - \tan \frac{v}{2}\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{\pi}{3} - 2\tan\frac{\pi}{12}.$$
If desired, one can also use a half angle formula to obtain $\tan \frac{\pi}{12} = 2 - \sqrt{3}$.

2. Let $f(x)$ be a strictly increasing non-negative continuous function. Calculate
$$\int_0^5 \frac{f(x)}{f(x)+f(5-x)}\,dx$$
and prove that your answer is correct.
Answer. Since $f(x) > f(0) \geq 0$ for $x > 0$, the denominator is never zero and the integral exists. The change of variables $u = 5 - x$ shows
$$\int_0^5 \frac{f(x)}{f(x)+f(5-x)}\,dx = -\int_5^0 \frac{f(5-u)}{f(5-u)+f(u)}\,du.$$
Renaming u as x and adding both sides shows that twice the integral we want is $\int_0^5 1\,dx = 5$, and so our original integral was $\frac{5}{2}$.

3. Find all real numbers satisfying the equation
$$(x-4)(x-1)x(x+2)(x+3)(x+6)+100 \;=\; 0.$$
Answer. Defining y to be $x+1$, the equation becomes
$$(y-5)(y-2)(y-1)(y+1)(y+2)(y+5)+100 \;=\; 0,$$

that is,

$$(y^2 - 25)(y^2 - 4)(y^2 - 1) + 100 \ = \ 0.$$

But

$$(y^2 - 25)(y^2 - 4)(y^2 - 1) + 100 \ = \ y^2(y^4 - 30y^2 + 129) \ = \ y^2((y^2 - 15)^2 - 96),$$

and the y satisfying $y^2((y^2-15)^2-96) = 0$ are 0, $\sqrt{15 \pm \sqrt{96}}$, and $-\sqrt{15 \pm \sqrt{96}}$. So there are five real numbers x satisfying

$$(x - 4)(x - 1)x(x + 2)(x + 3)(x + 6) + 100 \ = \ 0,$$

namley -1, $-1 + \sqrt{15 \pm \sqrt{96}}$, and $-1 - \sqrt{15 \pm \sqrt{96}}$.

4. A triangle of area $\frac{1}{2}$ lies in a unit square. Prove that at least two of its vertices are also vertices of the square.

   Answer. In fact for any rectangle (or parallelogram), the area of a triangle contained within it is at most half the area of the rectangle. To see this, observe

   (a) If not all vertices of the triangle lie on the boundary of the rectangle, we can shrink the sides of the rectangle to obtain an example where the vertices of the triangle do lie on the boundary of the rectangle, and the triangle:rectangle area ratio has increased. Hence it suffices to handle the case of all triangle vertices on the boundary of the rectangle.

   (b) Consider the case of two vertices of the triangle lying on one side of the rectangle. The area of the triangle is $\frac{1}{2}$ triangle base $\times$ triangle height. Triangle base is $\leq$ rectangle base, triangle height is $\leq$ rectangle height.So the only way the triangle area can reach half of rectangle area is for triangle base = rectangle base, triangle height = rectangle height. But triangle base = rectangle base means two vertices of the triangle are also rectangle vertices as to be shown.

   (c) The only other case is that the three triangle vertices lie on three different sides of the rectangle. Then two of the vertices lie on opposite sides. Consider the third vertex, and use a line through this vertex and perpendicular to the corresponding side of the rectangle to divide the rectangle and the triangle into two pieces. Each of these subtriangles has an entire edge on the boundary of its subrectangle, but must be shorter than the entire side of the subrectangle. So by the argument above in (b), the area of each subtriangle is less than half the corresponding subrectangle. That applies to the total as well, and so this case is inconsistent with the triangle having half the rectangle's area.

5. For each of the following three definitions of distance $d(\mathbf{a}, \mathbf{b})$ between points $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^2$ determine whether for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$, there exists $\mathbf{t} \in \mathbb{R}^2$ such that

$$d(\mathbf{x}, \mathbf{t}) + d(\mathbf{t}, \mathbf{y}) - d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{t}) + d(\mathbf{t}, \mathbf{z}) - d(\mathbf{y}, \mathbf{z}) = d(\mathbf{z}, \mathbf{t}) + d(\mathbf{t}, \mathbf{x}) - d(\mathbf{z}, \mathbf{x}) = 0.$$

   (i) $\quad d(\mathbf{a}, \mathbf{b}) \ := \ |a_1 - b_1| + |a_2 - b_2|,$

$$(ii) \quad d(\mathbf{a},\mathbf{b}) := \sqrt{(a_1-b_1)^2+(a_2-b_2)^2},$$

$$(iii) \quad d(\mathbf{a},\mathbf{b}) := \begin{cases} |a_1-b_1|+|a_2|+|b_2| & \text{if } a_1 \neq b_1 \\ |a_2-b_2| & \text{if } a_1 = b_1 \end{cases}.$$

Answer.
(i). Given points $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $\mathbf{z} = (z_1, z_2)$ in $\mathbb{R}^2$, define t_i to be the median of x_i, y_i and z_i, for $i = 1, 2$. Then $\mathbf{t} = (t_1, t_2)$ satisfies the displayed equation.

(ii). For $\mathbf{x}, \mathbf{y}, \mathbf{t} \in \mathbb{R}^2$ to satisfy $d(\mathbf{x}, \mathbf{t}) + d(\mathbf{t}, \mathbf{y}) - d(\mathbf{x}, \mathbf{y}) = 0$, it must be the case that $\mathbf{t}$ lies on the straight line from $\mathbf{x}$ to $\mathbf{y}$. So for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$ that are not co-linear, there does not exist $\mathbf{t} \in \mathbb{R}^2$ satisfying the displayed equation.

(iii). Suppose $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $\mathbf{z} = (z_1, z_2)$ are three points in $\mathbb{R}^2$. If $x_1 = y_1 = z_1$ then $\mathbf{t} = (t_1, t_2)$ defined by $t_1 = x_1$ and $t_2 = \text{median}(x_2, y_2, z_2)$ satisfies the displayed equation. Otherwise, $\mathbf{t} = (t_1, t_2)$ defined by $t_1 = \text{median}(x_1, y_1, z_1)$ and $t_2 = 0$ satisfies the displayed equation.

6. Show that the binomial coefficient
$$\binom{2n}{n}$$
is divisible by $n+1$ for any positive integer n.

   Answer.
$$\binom{2n}{n} = \binom{2n}{n-1}\left(\frac{n+1}{n}\right)$$
where the two binomial coefficients are integers. Since $gcd(n+1, n) = 1$, every prime power dividing n must also divide $\binom{2n}{n-1}$. Thus
$$\binom{2n}{n-1}\left(\frac{1}{n}\right)$$
is an integer and so $\binom{2n}{n}$ being $n+1$ times this integer is divisible by $n+1$.

7. Let $a_k = k$ for $1 \leq k \leq 4$ with $a_{k+4} = a_k$ for all $k \geq 1$. Find a closed form expression for
$$\Sigma_{k=1}^{\infty} a_k \frac{x^k}{k!}.$$
(This sum is $x + 2\frac{x^2}{2!} + 3\frac{x^3}{3!} + 4\frac{x^4}{4!} + 1\frac{x^5}{5!} + 2\frac{x^6}{6!} + \ldots$.) *Your answer may (but need not) involve complex numbers.*

   Answer. The Maclaurin series of $\sin x$, $\cos x$, $\sinh x$, and $\cosh x$ all have this kind of period 4 pattern and we can build the general period 4 pattern as a suitable linear combination.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \Sigma_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \Sigma_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

So

$$\frac{1}{2}(\sin x + \sinh x) = x + \frac{x^5}{5!} + \frac{x^9}{9!} + \ldots = \Sigma_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!}$$
$$\frac{1}{2}(-\sin x + \sinh x) = \frac{x^3}{3!} + \frac{x^7}{7!} + \ldots = \Sigma_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}.$$

And

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \Sigma_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \Sigma_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

with

$$\frac{1}{2}(\cos x + \cosh x) = 1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \ldots = \Sigma_{k=0}^{\infty} \frac{x^{4k}}{(4k)!}$$
$$\frac{1}{2}(-\cos x + \cosh x) = \frac{x^2}{2!} + \frac{x^6}{6!} + \ldots = \Sigma_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!}.$$

Thus our series is

$$\frac{1}{2}(\sin x + \sinh x) + (-\cos x + \cosh x) + \frac{3}{2}(-\sin x + \sinh x) + 2(\cos x + \cosh x - 2).$$

Remark 1: In terms of complex exponentials, we are combining the series for $e^{\omega x}$ as ω ranges among the fourth roots of one. Closed form representations for the period k version of this problem can be achieved using k'th roots of one.

Remark 2: One can also argue (as a student did) from the power series that the unknown function $f(x)$ satisfies the differential equation $f^{(iv)} = f$ and use appropriate initial conditions to obtain f.