Freshman Prize Exam

4:30 – 6:00 PM, March 14, 2007

Full proofs or explanations are expected on all answers.
Please write your netid on your exam booklet.

Problem 1) Calculate

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{x(2-x)}+1}\,dx.$$

Problem 2) Let $f(x)$ be a strictly increasing non-negative continuous function. Calculate

$$\int_0^5 \frac{f(x)}{f(x)+f(5-x)}\,dx$$

and prove that your answer is correct.

Problem 3) Find all real numbers satisfying the equation

$$(x-4)(x-1)x(x+2)(x+3)(x+6)+100 \;=\; 0.$$

Problem 4) A triangle of area $\frac{1}{2}$ lies in a unit square. Prove that at least two of its vertices are also vertices of the square.

Problem 5) For each of the following three definitions of distance $d(\mathbf{a},\mathbf{b})$ between points $\mathbf{a}=(a_1,a_2)$ and $\mathbf{b}=(b_1,b_2)$ in $\mathbb{R}^2$ determine whether for all $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^2$, there exists $\mathbf{t}\in\mathbb{R}^2$ such that

$$d(\mathbf{x},\mathbf{t})+d(\mathbf{t},\mathbf{y})-d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{t})+d(\mathbf{t},\mathbf{z})-d(\mathbf{y},\mathbf{z}) = d(\mathbf{z},\mathbf{t})+d(\mathbf{t},\mathbf{x})-d(\mathbf{z},\mathbf{x}) = 0.$$

(i) $\quad d(\mathbf{a},\mathbf{b}) \;:=\; |a_1-b_1|+|a_2-b_2|\,,$

(ii) $\quad d(\mathbf{a},\mathbf{b}) \;:=\; \sqrt{(a_1-b_1)^2+(a_2-b_2)^2},$

(iii) $\quad d(\mathbf{a},\mathbf{b}) \;:=\; \begin{cases} |a_1-b_1|+|a_2|+|b_2| & \text{if } a_1\neq b_1 \\ |a_2-b_2| & \text{if } a_1=b_1 \end{cases}.$

Problem 6) Show that the binomial coefficient

$$\binom{2n}{n}$$

is divisible by $n+1$ for any positive integer n.

Problem 7) Let $a_k = k$ for $1\le k\le 4$ with $a_{k+4}=a_k$ for all $k\ge 1$. Find a closed form expression for

$$\Sigma_{k=1}^{\infty} a_k \frac{x^k}{k!}.$$

(This sum is $x+2\frac{x^2}{2!}+3\frac{x^3}{3!}+4\frac{x^4}{4!}+1\frac{x^5}{5!}+2\frac{x^6}{6!}+\ldots.$) *Your answer may (but need not) involve complex numbers.*