Freshman Prize Exam 2006 Solutions

(1) Find the antiderivative:
\[ \int \frac{x^{11}}{\sqrt{x^6 - 1}} \, dx. \]

**Solution:** Let’s make the substitution \( u = x^6 - 1 \). Then we have that \( du = 6x^5 \, dx \) and the integral becomes
\[ \int \frac{x^{11}}{\sqrt{x^6 - 1}} \, dx = \int \frac{(u + 1) \, du/6}{\sqrt{u}} = \int \frac{1}{6} u^{1/2} \, du + \frac{1}{6} u^{-1/2} \, du = \]
\[ \frac{1}{6} \left( \frac{5}{2} u^{3/2} + \frac{1}{2} u^{1/2} \right) + C = \frac{1}{9} u^{3/2} + \frac{1}{3} u^{1/2} + C = \]
Substituting back \( u = x^6 - 1 \) we obtain:
\[ \int \frac{x^{11}}{\sqrt{x^6 - 1}} \, dx = \frac{1}{9} (x^6 + 2) \sqrt{x^6 - 1} + C \]
The integral can be computed using other substitutions like \( x^3 = \tan \theta \), \( x^3 = \cosh u \); or by integration by parts.

(2) Prove that the graph of a cubic polynomial \( y = x^3 + bx^2 + cx + d \) is rotationally symmetric about its point of inflection.

**Solution:** The second derivative is \( 6x + 2b \) and it is zero at \( x = -b/3 \). If we make the substitution \( \tilde{x} = x + b/3 \), which moves the inflection point to the \( y \)-axes, the equation of the cubic becomes
\[ y = \tilde{x}^3 + C \tilde{x} + D \]
where \( C \) and \( D \) are some constants. A second substitution \( \tilde{y} = y - D \) is needed to move the inflection point to the origin. In the new coordinates the equation of the cubic is
\[ \tilde{y} = \tilde{x}^3 + C \tilde{x} \]
which is rotationally symmetric because if \( (\tilde{x}, \tilde{y}) \) satisfies the above equation then the symmetric point \( (-\tilde{x}, -\tilde{y}) \) also satisfy the same equation.

(3) The sequence 1, 3, 4, 9, 10, 12, 13, … consists of all positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term in this sequence (where 1 is the first term, 3 is the second term, 4 is the third term…).

**Solution:** Numbers in this sequence are simply numbers who’s base 3 representation consists of only 1’s and 0’s. Since one hundred has a binary representation of 11000100, the hundredth term in the sequence must be \( 3^7 + 3^5 + 3^2 \).

(4) Suppose there are \( x \) socks in a drawer; some of them white some of them black. It is the case that when two socks are drawn without replacement, there is a probability of exactly \( 1/2 \) that either both are black or both are white. If \( x \) is at most 2006, what is the largest value \( x \) can take?

**Solution:** Let \( y \) be the number of black socks. The probability of picking two same colored socks is
\[ \frac{y(y-1)}{x(x-1)} + \frac{(x-y)(x-y-1)}{x(x-1)}. \]
Setting this equal to \( 1/2 \) and simplifying gives:
\[ x^2 - 4xy + 4y^2 - x = 0 \]
which gives $(x - 2y)^2 = x$ or equivalently that $y = \frac{x \pm \sqrt{x}}{2}$. Therefore $x$ must be a perfect square, the largest of which (below 2006) is 1936.

(5) For which real numbers $c$ is
\[ \frac{1}{2} (e^x + e^{-x}) \leq e^{cx^2} \]
for all real $x$?

Solution: Set $f(x) = e^{cx^2} - \frac{1}{2} (e^x + e^{-x})$. Note $f$ is infinitely differentiable with $f(0) = f'(0) = 0$ and $f''(0) = 2c - 1$. So $f(x) \geq 0$ for small $x$ implies $c \geq \frac{1}{2}$. We claim this condition on $c$ suffices to make $f$ positive for all $x$.

To see this, we use the everywhere convergent Maclaurin series of $e^x$ to obtain
\[ f(x) = \sum_{n=0}^{\infty} \frac{c^nx^{2n}}{n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \geq \sum_{n=0}^{\infty} \frac{c^nx^{2n}}{2^n n!} - \frac{1}{(2n)!} \]

For $n \geq 1$, we have $n + 1, n + 2, \ldots, 2n \geq 2$, and each coefficient of the above series is non-negative. Hence $f(x) \geq 0$ for all $x$.

(6) Let $Q$ be a quadrilateral of maximum area among all quadrilaterals with sides $a, b, c, \text{ and } d$.

a: Prove that $Q$ can be inscribed in a circle.

b: Show that the same maximum is obtained regardless of the order of the lengths around the perimeter of the quadrilateral.

Solution:

a: Assume the edges are ordered consecutively around the perimeter as $a, b, c, \text{ and } d$. Denote by $x$ the angle between adjacent edges $a$ and $b$. Similarly let $y$ be the angle between edges $c$ and $d$.

Consider the diagonal of the quadrilateral between the other two vertices of the quadrilateral. Applying the cosine law to the length of this diagonal gives
\[ c^2 + d^2 - 2cd \cos x = a^2 + b^2 - 2ab \cos y. \]

This equation implicitly defines $y$ as a function of $x$ with
\[ \frac{dy}{dx} = \frac{cd \sin x}{ab \sin y}. \]

The area of the quadrilateral is
\[ A(x) = \frac{cd \sin x}{2} + \frac{ab \sin y}{2} \]
and using $y'(x)$ from above, the critical point condition $A'(x) = 0$ gives
\[ 0 = \frac{cd \cos x}{2} + \frac{ab \cos y}{2} \cdot \left( \frac{cd \sin x}{ab \sin y} \right). \]

This equation then tells us $\cot x = -\cot y$ implying $x$ and $y$ are supplementary. Hence so are the other two opposite angles of the quadrilateral and it is inscribable in the circle.

b: If maximal area quadrilateral $ABCD$ has $AB = a, BC = b, CD = c, \text{ and } DC = d$, then by the above, $ABCD$ is inscribable in a circle. But then reflection in the perpendicular bisector of diagonal $AC$ applied to the points on the $B$ side of this diagonal preserves the circle and produces an inscribed quadrilateral of the same area whose order of sides is now $b, a, c,$
and $d$. Similarly any other two adjacent sides can be interchanged without affecting the maximal area, and the area does not depend on the order of the sides.

(7) Let $a$, $b$ and $c$ be integers whose greatest common divisor is 1. Show that there exist integers $m$ and $n$ such that $a + mc$ and $b + nc$ are relatively prime (i.e. have greatest common divisor 1.)

**Solution:** If $a = 0$ we can that $m = 1$ and $n = 0$ because

$$1 = (0, b, c) = (b, c).$$

Assume that $a \neq 0$ and let $\{p_i\}$ be all the primes which divide $a$. Let $\{q_j\}$ be all the primes among the $p_i$-es which do not divide $b$. We claim that the integers $a$ and $B = b + \prod q_j c$ are relatively prime.

Suppose that $p_i | (a, B)$ for some $i$. If $p_i$ divides $b$ than it also divides $B - b = \prod q_j c$. By construction $p_i$ is not equal to any of the $q_j$ therefore $p_i | c$, which contradicts the assumption that $(a, b, c) = 1$. On the other hand if $p_i$ does not divide $b$ then it is equal to one of the $q_j$-es and thus divides $\prod q_j c$; therefore it does not divide $B$.

In both case we have reach a contradiction. Thus $gcd(a, B)$ does not have any prime divisors and is equal to 1.