Freshman Prize Exam 2006 Solutions

(1) Find the antiderivative:

$$\int \frac{x^{11}}{\sqrt{x^6-1}}\,dx.$$

Solution: Let's make the substitution $u = x^6 - 1$. Then we have that $du = 6x^5dx$ and the integral becomes

$$\int \frac{x^{11}}{\sqrt{x^6-1}}\,dx = \int \frac{(u+1)\,du/6}{\sqrt{u}} = \int \frac{1}{6}u^{1/2} + \frac{1}{6}u^{-1/2}\,du =$$

$$\frac{1}{6}\frac{1}{3/2}u^{3/2} + \frac{1}{6}\frac{1}{1/2}u^{1/2} + C = \frac{1}{9}u^{3/2} + \frac{1}{3}u^{1/2} + C =$$

Substituting back $u = x^6 - 1$ we obtain:

$$\int \frac{x^{11}}{\sqrt{x^6-1}}\,dx = \frac{1}{9}(x^6+2)\sqrt{x^6-1} + C$$

The integral can be computed using other substitutions like $x^3 = \tan\theta$, $x^3 = \cosh u$; or by integration by parts.

(2) Prove that the graph of a cubic polynomial $y = x^3 + bx^2 + cx + d$ is rotationally symmetric about its point of inflection.
Solution: The second derivative is $6x + 2b$ and it is zero at $x = -b/3$. If we make the substitution $\tilde{x} = x + b/3$, which moves the inflection point to the y-axes, the equation of the cubic becomes

$$y = \tilde{x}^3 + C\tilde{x} + D$$

where C and D are some constants. A second substitution $\tilde{y} = y - D$ is needed to move the inflection point to the origin. In the new coordinates the equation of the cubic is

$$\tilde{y} = \tilde{x}^3 + C\tilde{x}$$

which is rotationally symmetric because if $(\tilde{x}, \tilde{y})$ satisfies the above equation then the symmetric point $(-\tilde{x}, -\tilde{y})$ also satisfy the same equation.

(3) The sequence $1, 3, 4, 9, 10, 12, 13, \ldots$ consists of all positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term in this sequence (where 1 is the first term, 3 is the second term, 4 is the third term...).
Solution: Numbers in this sequence are simply numbers who's base 3 representation consists of only 1's and 0's. Since one hundred has a binary representation of 11000100, the hundredth term in the sequence must be $3^7 + 3^6 + 3^2$.

(4) Suppose there are x socks in a drawer; some of them white some of them black. It is the case that when two socks are drawn without replacement, there is a probability of exactly 1/2 that either both are black or both are white. If x is at most 2006, what is the largest value x can take?
Solution: Let y be the number of black socks. The probability of picking two same colored socks is

$$\frac{y(y-1)}{x(x-1)} + \frac{(x-y)(x-y-1)}{x(x-1)}.$$

Setting this equal to 1/2 and simplifying gives:

$$x^2 - 4xy + 4y^2 - x = 0$$

which gives $(x-2y)^2 = x$ or equivalently that $y = \frac{x \pm \sqrt{x}}{2}$. Therefore x must be a perfect square, the largest of which (below 2006) is 1936.

(5) For which real numbers c is

$$\frac{1}{2}\left(e^x + e^{-x}\right) \leq e^{cx^2}$$

for all real x?

Solution: Set $f(x) = e^{cx^2} - \frac{1}{2}\left(e^x + e^{-x}\right)$. Note f is infinitely differentiable with $f(0) = f'(0) = 0$ and $f''(0) = 2c - 1$. So $f(x) \geq 0$ for small x implies $c \geq \frac{1}{2}$. We claim this condition on c suffices to make f positive for all x.

To see this, we use the everywhere convergent Maclaurin series of e^x to obtain

$$f(x) = \Sigma_{n=0}^{\infty} \frac{c^n x^{2n}}{n!} - \Sigma_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \geq \Sigma_{n=0}^{\infty} x^{2n}\left(\frac{1}{2^n n!} - \frac{1}{(2n)!}\right).$$

For $n \geq 1$, we have $n+1, n+2, \ldots, 2n \geq 2$, and each cofficient of the above series is non-negative. Hence $f(x) \geq 0$ for all x.

(6) Let Q be a quadrilateral of maximum area among all quadrilaterals with sides a, b, c, and d.

a: Prove that Q can be inscribed in a circle.

b: Show that the same maximum is obtained regardless of the order of the lengths around the perimeter of the quadrilateral.

Solution:

a: Assume the edges are ordered consecutively around the perimeter as a, b, c, and d. Denote by x the angle between adjacent edges a and b. Similarly let y be the angle between edges c and d.

Consider the diagonal of the quadrilateral between the other two vertices of the quadrilateral. Applying the cosine law to the length of this diagonal gives

$$c^2 + d^2 - 2cd\cos x = a^2 + b^2 - 2ab\cos y.$$

This equation implicitly defines y as a function of x with

$$\frac{dy}{dx} = \frac{cd\sin x}{ab\sin y}.$$

The area of the quadrilateral is

$$A(x) = \frac{cd\sin x}{2} + \frac{ab\sin y}{2}$$

and using $y'(x)$ from above, the crititcal point conditon A'(x)=0 gives

$$0 = \frac{cd\cos x}{2} + \frac{ab\cos y}{2}\left(\frac{cd\sin x}{ab\sin y}\right).$$

This equation then tells us $\cot x = -\cot y$ implying x and y are supplementary. Hence so are the other two opposite angles of the quadrilateral and it is inscribable in the circle.

b: If maximal area quadrilateral ABCD has $AB = a$, $BC = b$, $CD = c$, and $DC = d$, then by the above, ABCD is inscribable in a circle. But then reflection in the perpendicular bisector of diagonal AC applied to the points on the B side of this diagonal preserves the circle and produces an inscribed quadrilateral of the same area whose order of sides is now b, a, c, and d. Similarly any other two adjacent sides can be interchanged without affecting the maximal area, and the area does not depend on the order of the sides.

(7) Let a, b and c be integers whose greatest common divisor is 1. Show that there exist integers m and n such that $a + mc$ and $b + nc$ are relatively prime (i.e. have greatest common divisor 1.)
Solution: If $a = 0$ we can that $m = 1$ and $n = 0$ because

$$1 = (0, b, c) = (b, c).$$

Assume that $a \neq 0$ and let $\{p_i\}$ be all the primes which divide a. Let $\{q_j\}$ be all the primes among the p_i-es which do not divide b. We claim that the integers a and $B = b + \prod q_j c$ are relatively prime.

Suppose that $p_i|(a, B)$ for some i. If p_i divides b than it also divides $B - b = \prod q_j c$. By construction p_i is not equal to any of the q_j therefore $p_i|c$, which contradicts the assumption that $(a, b, c) = 1$. On the other hand if p_i does not divide b then it is equal to one of the q_j-es and thus divides $\prod q_j c$; therefore it does not divide B.

In both case we have reach a contradiction. Thus $gcd(a, B)$ does not have any prime divisors and is equal to 1.