

Freshman Prize Exam 2006

Full proofs or explanations are expected on all answers.

Please write your netid on your exam booklet.

(1) Find the antiderivative:

$$\int \frac{x^{11}}{\sqrt{x^6 - 1}} dx.$$

(2) Prove that the graph of a cubic polynomial $y = x^3 + bx^2 + cx + d$ is rotationally symmetric about its point of inflection.

(3) The sequence 1, 3, 4, 9, 10, 12, 13, ... consists of all positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term in this sequence (where 1 is the first term, 3 is the second term, 4 is the third term...).

(4) Suppose there are x socks in a drawer; some of them white some of them black. It is the case that when two socks are drawn without replacement, there is a probability of exactly $1/2$ that either both are black or both are white. If x is at most 2006, what is the largest value x can take?

(5) For which real numbers c is

$$\frac{1}{2}(e^x + e^{-x}) \leq e^{cx^2}$$

for all real x ?

(6) Let Q be a quadrilateral of maximum area among all quadrilaterals with sides a , b , c , and d .

a) Prove that Q can be inscribed in a circle.

b) Show that the same maximum is obtained regardless of the order of the lengths around the perimeter of the quadrilateral.

(7) Let a , b and c be integers whose greatest common divisor is 1. Show that there exist integers m and n such that $a + mc$ and $b + nc$ are relatively prime (i.e. have greatest common divisor 1.)