Freshman Prize Exam 2006

Full proofs or explanations are expected on all answers.

Please write your netid on your exam booklet.

(1) Find the antiderivative:
\[ \int \frac{x^{11}}{\sqrt{x^6 - 1}} \, dx. \]

(2) Prove that the graph of a cubic polynomial \( y = x^3 + bx^2 + cx + d \) is rotationally symmetric about its point of inflection.

(3) The sequence 1, 3, 4, 9, 10, 12, 13, \ldots consists of all positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term in this sequence (where 1 is the first term, 3 is the second term, 4 is the third term...).

(4) Suppose there are \( x \) socks in a drawer; some of them white some of them black. It is the case that when two socks are drawn without replacement, there is a probability of exactly \( \frac{1}{2} \) that either both are black or both are white. If \( x \) is at most 2006, what is the largest value \( x \) can take?

(5) For which real numbers \( c \) is
\[ \frac{1}{2} (e^x + e^{-x}) \leq e^{cx^2} \]
for all real \( x \)?

(6) Let \( Q \) be a quadrilateral of maximum area among all quadrilaterals with sides \( a, b, c, \) and \( d \).
   a) Prove that \( Q \) can be inscribed in a circle.
   b) Show that the same maximum is obtained regardless of the order of the lengths around the perimeter of the quadrilateral.

(7) Let \( a, b \) and \( c \) be integers whose greatest common divisor is 1. Show that there exist integers \( m \) and \( n \) such that \( a + mc \) and \( b + nc \) are relatively prime (i.e. have greatest common divisor 1.).