Honors Introduction to Analysis I
Homework V

Solution

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Problem 1 *Let A be an open set. Show that if a finite number of points are removed from A, the remaining set is still open. Is the same true if a countable number of points are removed?*

Solution. If we order the points removed from A, $x_1 < x_2 < ... < x_k$ we can write

$$A - \{x_1, x_2, ..., x_k\} = A \cap ((-\infty, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup ...(x_{k-1}, x_k) \cup (x_k, \infty))$$

$B := (-\infty, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup ...(x_{k-1}, x_k) \cup (x_k, \infty)$ is a union of open sets, hence open. $A \cap B$ is a finite intersection of open sets, hence it is also open.

The statement is not true for countable number of points. Consider $\mathbb{R} - \mathbb{Q}$. This is not open, since given any irrational point x, by the density of rational, we can find for every n a rational $y \in (x - 1/n, x + 1/n)$, hence we couldn't find a neighborhood around x completely in $\mathbb{R} - \mathbb{Q}$. □

Problem 2 *Give an example of a set A that is not closed but such that every point of A is a limit-point.*

Solution. Consider the open interval $A = (0, 1)$. For any $x \in (0, 1)$ there is an open interval $(x - 1/n, x + 1/n)$ for some $n \geq N$ such that $(x - 1/n, x + 1/n) \subset (0, 1)$. But in $(x - 1/n, x + 1/n)$ there are infintely points from $(0, 1)$, hence x is a limit point. On the other hand 0 and 1 are limit points, but they are not in A, hence A is not closed. □

Problem 3 *Show that the set of limit-points of a sequence is a closed set.*

Solution. Let L be the set of limit points of the sequence $\{x_n\}$. Let a be a limit point of the set L. Then for all errors $1/n$, there exists an $x \in L$ such that $|x - a| < \frac{1}{2n}, x \neq a$. But x is a limit point of the sequence, hence for all n there exist infinitely many x_k's such that $|x - x_k| < \frac{1}{2n}$.

From the above we have that for all $n \in \mathbb{N}$, $|a - x_k| = |a - x + x - x_k| < |a - x| + |x - x_k| < 1/2n + 1/2n < 1/n$ for infinitely many x_k's. Hence a is also a limit point of the sequence, so $a \in L$. Therefore L is closed. □

Problem 4 *Show that the Cantor set is a perfect set that contains no open intervals. Show that it is uncountable.*

Solution. Let's denote with C_n the set obtain after we have taken out the middle thirds n times. So $C_n = [0, 1/3^n] \cup [2/3^n, 3/3^n] \cup ... \cup [(3^n - 1)/3^n, 1]$ (there are 2^n intervals). Since C_n is a finite union of closed intervals, it is a closed set. By definition the Cantor set $C = \bigcap_{n \in \mathbb{N}} C_n$, so it's a countable intersection of closed sets, hence it is closed.

Given any $x \in C$, then $x \in C_n$, for any n. In particular x belongs to one of the 2^n intervals of C_n. Consider $x_n \in C$ to be the closest endpoint to x of this interval, such that $x_n \neq x$ (if x is itself an endpoint, choose x_n to be the other endpoint). We have that $|x - x_n| \leq 1/3^n < 1/n$. Given any n we can find at least one $x_n \in C$ (the endpoints of the intervals are in C), such that $|x - x_n| < 1/n$ and $x_n \neq x$. Hence x is a limit point. As C is closed, this means that C is perfect.

Assume C contains an open interval (a, b). Then there is an n (by the Archemedean property) such that $|a - b| > 1/3^n$. But after the n^{th} step, all the remaining intervals in C_n have length $1/3^n$. Hence there is a middle-third taken out of (a,b), which means that there are points in (a,b), that are not in C. Hence $(a,b) - C \neq \emptyset$ (contradiction). So C doesn't contain any open interval.

To see that C is uncountable, either use the fact that there is a bijection $C \to [0,1]$ (which was used in the first HW), or use the fact that the Cantor set consists of all numbers expressible with just 0's and 2's in base 3. This means that we can build a bijection between C and the set of sequences consisting of 0 and 2, which has the same cardinality as $2^{\mathbb{N}}$, which is uncountable. □

Problem 5 *Show that a union of open intervals can be written as a disjoint union of open intervals.*

Solution. The whole problem can be reduced to writing the union of two intervals as a union of disjoint ones (in case we have several, or uncountably many, we do induction). We are given (a,b) and (c,d) and we want to write $(a,b) \cup (c,d)$ as a union of disjoint intervals. There are three cases that can occur:

- $a < b \leq c < d$. Then the two intervals are already disjoint, so we are done.
- $a \leq c < b \leq d$. Then $(a,b) \cup (c,d) = (a,d)$.
- $a \leq c < d \leq b$. Then $(a,b) \cup (c,d) = (a,b)$.

□

Problem 6 *Show that an open set cannot be written in two different ways as a disjoint union of open intervals (except for a change in the order of the intervals).*

Solution. Assume that we can write the open set in two different ways. Then $D = \bigcup A_i = \bigcup B_k$ with at least one $A_i \neq B_k$. We will show that any A_i and B_k are either disjoint, or the same.

Assume not, i.e. $A_i \setminus B_k \neq \emptyset$ for some i,k ($A_i = (a,b)$ and $B_k = (c,d)$). W.l.o.g. assume $a < c$, so $A_i \setminus B_k = (a,c]$. The point c is in A_i, hence it is in $D = \bigcup B_k$. c must belong to some open B_m disjoint from B_k, so there is another neighborhood $(c - 1/n, c + 1/n) \subset B_m$. But $(c, c - 1/n) \subset B_k$, so we get that $B_k \cap B_m \neq \emptyset$ (contradiction).

In the same way we get that $B_k \setminus A_i = \emptyset$ for all k,i. This means that given any $x \in A_i$ for some i, then $x \in B_k$ for some k, hence $A_i \subseteq B_k$. But since $B_k \setminus A_i = \emptyset$, then $A_i = B_k$. So except for a change in the order of the intervals, there is a unique way or writing any open set as a disjoint union of open intervals. □

Problem 7 *Show that compact sets are closed under arbitrary intersections and finite unions.*

Solution. Compact sets are closed and bounded. Arbitrary intersections and finite unions of closed sets are closed. We just need to prove this is true for boundedness.

If $C = \bigcap C_i$, and each C_i is bounded by M_i (which is a finite number), then let $M = \max M_i$ (it exists since all M_i are finite). Given any $x \in C$, $x \in C_i$ for all i. Then $|x| \leq M_i \leq M$, hence C is bounded. So arbitrary intersections of bounded sets are bounded.

If $C = \bigcup_{n=1}^{k} C_n$, and each C_n is bounded by M_n, consider $M = \max M_n$. Given any $x \in C$ then x is in some C_n, for some n. But then $|x| \leq M_n \leq M$, hence C is also bounded. So finite unions of bounded sets are bounded. □

Problem 8 *If $B_1, ..., B_n$ is a finite open cover of a compact set A, can the union $B_1 \cup ... \cup B_n$ equal A exactly?*

Solution. If $B_1 \cup ... \cup B_n = A$, then A is a union of open sets, hence it is open. Since A is also compact, it has to be closed and bounded. This means that A is closed and open (clopen). But the only clopen sets in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$ itself (you should prove this). But $\mathbb{R}$ is not compact. Hence $A = \emptyset$. Hence the statement is true only for the empty set; for any other compact set it is false. □

Problem 9 *For two non-empty sets of numbers A and B, define $A + B$ to be the set of all sums $a + b$ where a is in A and b is in B. Show that if A is open, then $A + B$ is open. Show that if A and B are compact, then $A + B$ is compact. Give an example where A and B are closed but $A + B$ is not.*

Solution. Let $x \in A + B$. Then $x = a + b$ for some $a \in A$ and $b \in B$. As A is open, there exists an open interval $(a - 1/n, a + 1/n) \subset A$ around a. But then the interval $(a + b - 1/n, a + b + 1/n)$ is contained in $A + B$ and $x \in (a + b - 1/n, a + b + 1/n) \subset A + B$. Hence $A + B$ is open.

Let $\{x_n\}$ be a sequence of numbers belonging to $A + B$. Then each $x_n = a_n + b_n$ for some $a_n \in A$ and $b_n \in B$. The sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to $a \in A$ since A is compact. Now the subsequence $\{b_{n_k}\}$, being itself a sequence in B, has a converging subsequence $\{b_{n_{k_l}}\}$ to some b in B, since B is also compact. This means that the sub(sub)sequence $\{x_{n_{k_l}}\} = \{a_{n_{k_l}}\} + \{b_{n_{k_l}}\}$ converges to $a + b \in A + B$. Hence $A + B$ is compact.

For the example, consider $A = \{-n | n = 1, 2, ...\}$ and $B = \left\{n + \frac{1}{n} | n = 1, 2, ...\right\}$. Since they are discrete sets, they are closed. Now their sum doesn't include the number 0, but 0 is a limit point, because the sequence $\{1/n\} \in A + B$ converges to it, so $A + B$ is not closed. □

Problem 10 *If A is compact, show that* $\sup A$ *and* $\inf A$ *belong to A. Give an example of a non-compact set A such that both* $\sup A$ *and* $\inf A$ *belong to A.*

Solution. Since A is compact, it is closed and bounded. Since it is bounded both $\sup A$ and $\inf A$ are finite.

Choose b not in A, such that b greater than all elements of A (it exists, since A is bounded). Choose also an element $a \in A$. Now, we will divide and conquer. Let $a_1 = a$ and $b_1 = b$. If $(a+b)/2 \in A$, define $a_2 = (a+b)/2$ and $b_2 = b$. If $(a+b)/2$ is an upper bound of A, define $a_2 = a$ and $b_2 = (a+b)/2$. Continue this way recursively, and we obtain two monotonuous sequences $\{a_n\}$ and $\{b_n\}$ such that $a_{n+1} \geq a_n$ (increasing) and $b_{n+1} \leq b_n$ (decreasing). Since they are bounded and monotonuous, they will converge. By the construction $a_n \leq \sup A \leq b_n$. Passing to the limits we get: $\lim_{n\to\infty} a_n \leq \sup A \leq \lim_{n\to\infty} b_n$. Hence $\lim a_n = \sup A$, and since A is closed, and $\sup A$ is a limit point, we get that $\sup A \in A$. The proof for $\inf A$ is similar (just change the direction of the inequalities).

For the example of the non-compact set, consider an "almost closed set" $A = [0, 1) \cup (1, 2]$. This is not closed (1 is a limit point, but it is not contained in the set), hence it is not compact. But $\sup A = 0$ and $\inf A = 2$ are both elements of A. □