Final Exam (8 problems)

(1) Suppose that $\{A_n\}$ is a sequence of events. Show that $P(\limsup A_n) \geq \limsup P(A_n)$.

(2) Let X be an integer-valued random variable with characteristic function $\phi(t)$. (a) Prove the inversion formula:

$$P(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \phi(t)\, dt.$$

(b) Consider the random walk $S_n = X_1 + \cdots + X_n$, where the X_i are integer-valued, symmetric, and satisfy $E|X_i| < \infty$. Show that $P(S_n = 0 \text{ i.o.}) = 1$. (You may use without proof that $P(S_n = 0 \text{ i.o.}) = 1$ iff $\sum_{n\geq 0} P(S_n = 0) = \infty$. We proved this in class when $X_k = \pm 1$, and the proof generalizes easily.)

(3) An alternative way to define (as opposed to the way we did it in class with exponential random variables) continuous-time Markov chains is shown by the following example. A birth chain is a positive interger-valued *Markov* process where

$$P(X_{t+\epsilon} = n + 1 | X_t = n) = \beta_n \epsilon + o(\epsilon)$$

for positive constants $\{\beta_k\}$. Assume $X_0 = 1$.

(a) For arbitrary $\beta_i, \beta_1 \neq \beta_2$, find $P(X_t = 1)$ and $P(X_t = 2)$ and compare these to the probabilities you would expect from a construction with exponential random variables.

(b) When $\beta_k = k$ use induction to show that $P(X_t = n) = e^{-t}(1 - e^{-t})^n$.

(4) Let $\phi(x) = \frac{1}{x} - [\frac{1}{x}]$ ($[\cdot]$ is the greatest integer function). Note that $\phi : (0, 1) \to (0, 1)$. Show that ϕ preserves measure with density $(\log 2)^{-1} \frac{1}{1+x}$ with respect to Lebesgue measure. (Hint: graph ϕ.)

(5) Prove that ϕ is ergodic if the following property holds:

For every f, g belonging to a vector space $\mathcal{H}$ which is dense in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$,

$$\lim_{n\to\infty} E[f(g \circ \phi^n)] = E(f)E(g).$$

Here ϕ^n is the n-fold composition.

(6) Let $B(t)$ be standard Brownian motion starting at 0 and define $M = \sup_{t>0}[B(t) - t]$. Use the Strong Markov Property to show that M has an exponential distribution. Be sure to explicitly give the Y you are using in the SMP. (Hint: Let $\tau_a = \inf_t\{B(t) - t = a\}$ and show that $P(\tau_{a+b} < \infty | \tau_a < \infty) = P(\tau_b < \infty)$.)

(7) Let $B(t)$ be standard Brownian motion starting at 0.

(a) Show that $B^4(t) - 6B^2(t) + 3t^2$ is a martingale.

(b) Show that if τ is a stopping time with $E\tau < \infty$ then $E\tau^2 \le EB^4(\tau)$.

(c) Let $\tau = \inf_t\{|B(t)| = a\}, a > 0$. Show that $E\tau^2 = \frac{5}{3}a^4$.

(8) Suppose $\{X(t), t \ge 0\}$ is a Levy Process and that $X(t)$ has a symmetric stable law with $\alpha \in (0, 2]$. In particular,

$$Ee^{iuX(t)} = e^{-t|u|^\alpha}.$$

Assume that with probability one, the paths are cadlag.

(a) Show that for each fixed t, $X(s)$ is a.s. continuous at $s = t$.

(b) Express

$$P\left(\max_{1\le k\le n}\left|X(\frac{k}{n}) - X(\frac{k-1}{n})\right| \ge \epsilon\right)$$

in terms of the distribution of $X(1)$.

(c) Using (b) show that $P(X(t)$ is continuous on $[0, 1]) = 0$.