Bridges of Random Walks in a Random Environment

Jonathon Peterson

Cornell University
Department of Mathematics

Joint work with Nina Gantert

February 25, 2010
$S_n$ simple random walk.

**Question**

What does the path of the random walk look like conditioned on $\{S_{2n} = 0\}$?
Introduction

Bridges of Random Walks

$S_n$ simple random walk.

$$
\begin{array}{c}
1 - p \\
x - 1
\end{array} \quad \begin{array}{c}
\quad \\
x
\end{array} \quad \begin{array}{c}
p \\
x + 1
\end{array}
$$

Question

What does the path of the random walk look like conditioned on \( \{S_{2n} = 0\} \)?

Distribution doesn’t depend on \( p \).

Scaled by \( \sqrt{n} \), converges to Brownian Bridge.
RWRE in $\mathbb{Z}$ with i.i.d. environment

An environment $\omega = \{\omega_x\}_{x \in \mathbb{Z}} \in \Omega = [0, 1]^\mathbb{Z}$. $P$ an i.i.d. product measure on $\Omega$. 
RWRE in $\mathbb{Z}$ with i.i.d. environment

An environment $\omega = \{\omega_x\}_{x \in \mathbb{Z}} \in \Omega = [0, 1]^\mathbb{Z}$. $P$ an i.i.d. product measure on $\Omega$.

Quenched law $P_\omega$: fix an environment. $X_n$ a random walk: $X_0 = 0$, and

$$P_\omega(X_{n+1} = y + 1 | X_n = y) := \omega_y$$

$$1 - \omega_x \quad \omega_x$$

$x - 1 \quad x \quad x + 1$
RWRE in $\mathbb{Z}$ with i.i.d. environment

An environment $\omega = \{\omega_x\}_{x \in \mathbb{Z}} \in \Omega = [0, 1]^\mathbb{Z}$. $P$ an i.i.d. product measure on $\Omega$.

**Quenched law** $P_\omega$: fix an environment. $X_n$ a random walk: $X_0 = 0$, and

$$P_\omega(X_{n+1} = y + 1|X_n = y) := \omega_y$$

![Diagram](attachment:diagram.png)

**Averaged law** $\mathbb{P}$: average over environments.

$$\mathbb{P}(G) := \int_\Omega P_\omega(G) dP(\omega)$$
Transience Criterion

A crucial statistic is:

$$\rho_x := \frac{1 - \omega_x}{\omega_x}$$

**Theorem (Solomon ’75)**

1. If $E_P(\log \rho_0) < 0$ then, $\lim_{n \to \infty} X_n = +\infty$, $\mathbb{P} - a.s.$
2. If $E_P(\log \rho_0) > 0$ then, $\lim_{n \to \infty} X_n = -\infty$, $\mathbb{P} - a.s.$
3. If $E_P(\log \rho_0) = 0$ then, $X_n$ is recurrent.
Bridges of RWRE

Assume i.i.d. environments.
Assume $E_P \log \rho < 0$ (transient to $+\infty$).

Question

What does the path of the random walk look like conditioned on $\{S_{2n} = 0\}$?

- Does it depend on the environment?
- Does it look like a Brownian Bridge?
- Is the right scaling $\sqrt{n}$?
Scale Parameter $\kappa(P)$ for Transient RWRE

Assume $E_P(\log \rho) < 0$ (transience to the right).

Define $\kappa = \kappa(P)$ by

$$E_P \rho^\kappa = 1.$$

$\kappa$ is related to the strength of the “traps”.

$E_P[\rho^p]$
Relevance of $\kappa$

**Theorem (Law of Large Numbers - Solomon ’75)**

$$\lim_{n \to \infty} \frac{X_n}{n} = v, \text{ and } v > 0 \iff \kappa > 1.$$

**Theorem (Averaged Limit Laws - Kesten, Kozlov, Spitzer ’75)**

\[(a) \quad \kappa \in (0, 1) \Rightarrow \lim_{n \to \infty} \mathbb{P}\left(\frac{X_n}{n^{\kappa}} \leq x\right) = 1 - L_{\kappa, b}(x^{-1/\kappa})\]

\[(b) \quad \kappa \in (1, 2) \Rightarrow \lim_{n \to \infty} \mathbb{P}\left(\frac{X_n - n\nu_P}{n^{1/\kappa}} \leq x\right) = 1 - L_{\kappa, b}(-x)\]

\[(c) \quad \kappa > 2 \Rightarrow \lim_{n \to \infty} \mathbb{P}\left(\frac{X_n - n\nu_P}{b\sqrt{n}} \leq x\right) = \Phi(x)\]

where $L_{\kappa, b}$ is an $\kappa$-stable distribution function.
Case I: Positive and Negative Drifts

\[ \omega_{\min} := \inf\{ t : P(\omega_0 \leq t) > 0 \}. \]

\[ E_P[\log \rho] < 0 \text{ and } \omega_{\min} < \frac{1}{2} \Rightarrow \kappa \in (0, \infty). \]
**Case I: Positive and Negative Drifts**

\[ \omega_{\text{min}} := \inf \{ t : P(\omega_0 \leq t) > 0 \}. \]

\[ E_P[\log \rho] < 0 \text{ and } \omega_{\text{min}} < 1/2 \implies \kappa \in (0, \infty). \]

**Theorem (G. P. - ’09)**

Assume \( E_P \log \rho < 0 \) and \( \omega_{\text{min}} < 1/2 \). Then,

\[
\lim_{n \to \infty} P_{\omega} \left( \max_{k \leq 2n} |X_k| \leq n^\beta \ \bigg| \ X_{2n} = 0 \right) = \begin{cases} 0 & \beta < \frac{\kappa}{\kappa+1} \\ 1 & \beta > \frac{\kappa}{\kappa+1} \end{cases}, \quad \text{P - a.s.}
\]

Suggests scaling of \( n^{\kappa/(\kappa+1)} \).

- \( \kappa < 1 \) - Subdiffusive scaling
- \( \kappa > 1 \) - Superdiffusive scaling
Case II: Positive and Zero Drifts

Theorem (G. P. - ’09)
Assume $\omega_{\min} = 1/2$ and $P(\omega_0 = 1/2) \in (0, 1)$. Then, for any $\varepsilon > 0$

$$\lim_{n \to \infty} P_{\omega} \left( \max_{k \leq 2n} |X_k| \leq n^{1-\varepsilon} \mid X_{2n} = 0 \right) = 0, \quad P - a.s.$$ 

and

$$\lim_{n \to \infty} P_{\omega} \left( \max_{k \leq 2n} |X_k| \leq \frac{n}{(\log n)^{2-\varepsilon}} \mid X_{2n} = 0 \right) = 1, \quad P - a.s.$$ 

We suspect the proper scaling is $\frac{n}{(\log n)^2}$. 
Case III: Strictly Positive Drifts

Theorem (G. P. ’09)

Assume $\omega_{\text{min}} > 1/2$, $P(\omega_0 = \omega_{\text{min}}) \in (0, 1)$ and $P(\omega_0 \in (\omega_{\text{min}}, \omega_{\text{min}} + \delta)) = 1$ for some $\delta > 0$. Then, for any $\varepsilon > 0$

$$\lim_{n \to \infty} P_{\omega} \left( \max_{k \leq 2n} |X_k| \leq n^{1-\varepsilon} \Bigg| X_{2n} = 0 \right) = 0, \quad P - \text{a.s.}$$

and

$$\lim_{n \to \infty} P_{\omega} \left( \max_{k \leq 2n} |X_k| \leq \frac{n}{(\log n)^{2-\varepsilon}} \Bigg| X_{2n} = 0 \right) = 1, \quad P - \text{a.s.}$$

Again, we suspect the proper scaling is $\frac{n}{(\log n)^2}$.
Decay of $P_\omega(X_{2n} = 0)$

Case I: $\omega_{\text{min}} < 1/2$

$$P_\omega(X_{2n} = 0) = \exp \left\{ -n^{\frac{\kappa}{\kappa+1}} + o(1) \right\}$$
Decay of $P_\omega(X_{2n} = 0)$

Case I: $\omega_{\min} < 1/2$

$$P_\omega(X_{2n} = 0) = \exp \left\{ -n^{\frac{\kappa}{\kappa + 1}} + o(1) \right\}$$

Case II: $\omega_{\min} = 1/2$, and $\alpha = P(\omega_0 = 1/2) \in (0, 1)$.

$$P_\omega(X_{2n} = 0) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{\pi \log \alpha}{4} + o(1) \right) \right\}$$

where $I(0) = -\frac{1}{2} \log(\frac{4}{1 - \omega_{\min}})$.
Decay of $P_\omega(X_{2n} = 0)$

Case I: $\omega_{\text{min}} < 1/2$

$$P_\omega(X_{2n} = 0) = \exp \left\{ -n^{\kappa \kappa + 1} + o(1) \right\}$$

Case II: $\omega_{\text{min}} = 1/2$, and $\alpha = P(\omega_0 = 1/2) \in (0, 1)$.

$$P_\omega(X_{2n} = 0) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{|\pi \log \alpha|^2}{4} + o(1) \right) \right\}$$

Case III: $\omega_{\text{min}} > 1/2$, and $\alpha = P(\omega_0 = \omega_{\text{min}}) \in (0, 1)$

$$P_\omega(X_{2n} = 0) = \exp \left\{ -I(0)n \right\}$$

where $I(0) = -\frac{1}{2} \log(4\omega_{\text{min}}(1 - \omega_{\text{min}}))$. 
Decay of $P_\omega(X_{2n} = 0)$

Case I: $\omega_{\text{min}} < 1/2$

$$P_\omega(X_{2n} = 0) = \exp\left\{-n^{\frac{\kappa}{\kappa+1}} + o(1)\right\}$$

Case II: $\omega_{\text{min}} = 1/2$, and $\alpha = P(\omega_0 = 1/2) \in (0, 1)$.

$$P_\omega(X_{2n} = 0) = \exp\left\{-\frac{n}{(\log n)^2} \left(\frac{|\pi \log \alpha|^2}{4} + o(1)\right)\right\}$$

Case III: $\omega_{\text{min}} > 1/2$, and $\alpha = P(\omega_0 = \omega_{\text{min}}) \in (0, 1)$

$$P_\omega(X_{2n} = 0) = \exp\left\{-I(0)n - \frac{n}{(\log n)^2} \left(|\pi \log \alpha|^2 + o(1)\right)\right\}$$

where $I(0) = -\frac{1}{2} \log(4\omega_{\text{min}}(1 - \omega_{\text{min}}))$. 
Potential and Traps

\[ V(i) := \begin{cases} 
\sum_{k=0}^{i-1} \log \rho_k, & i > 0 \\
0, & i = 0 \\
\sum_{k=i}^{-1} - \log \rho_k, & i < 0 
\end{cases} \]

**Trap:** Atypical section where the potential is increasing.
Zero Speed RWRE: $P(w=3/4)=0.55$, $P(w=1/3)=0.45$
Escaping Traps

Probability of escaping a trap of Height $H$. 

$$P_{\omega}(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}}$$

Time to escape trap of height $H$ 

$$\approx \exp(e^{-H})$$
**Escaping Traps**

Probability of escaping a trap of height $H$.

$$P_\omega(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}} \approx e^{-V(b)} = e^{-H}$$

---

**Graph:**

- The graph shows a function $V(b)$ with a peak at $V(b) = H$.
- The x-axis represents $b$, ranging from $-1$ to $0$, with $b$ labeled on the right side of the graph.
- The y-axis is not explicitly labeled but represents the values of the function $V(b)$.
Escaping Traps

Probability of escaping a trap of Height $H$.

$$P_\omega(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}} \approx e^{-V(b)} = e^{-H}$$

Time to escape trap of height $H \approx \text{Exp}(e^{-H})$. 

![Graph showing the relationship between $V(b)$ and $H$]
Scaling of $T_n$

How long does the largest trap in $[0, n]$ contain the walk?

![Graph showing potential $V(x)$ versus location $x$.]
Scaling of $T_n$

How long does the largest trap in $[0, n]$ contain the walk?

- Time to escape trap of height $H \approx \text{Exp}(e^{-H})$. 

![Graph showing potential $V(x)$ over location $x$]
Scaling of $T_n$

How long does the largest trap in $[0, n]$ contain the walk?

- Time to escape trap of height $H \approx Exp(e^{-H})$.
- Largest uphill of $V(\cdot)$ in $[0, n]$ is $\sim \frac{1}{\kappa} \log n$ (Erdös & Renyi ’70).
Scaling of $T_n$

How long does the largest trap in $[0, n]$ contain the walk?

- Time to escape trap of height $H \approx \text{Exp}(e^{-H})$.
- Largest uphill of $V(\cdot)$ in $[0, n]$ is $\sim \frac{1}{\kappa} \log n$ (Erdős & Renyi ’70).

$\Rightarrow$ scaling of $n^{1/\kappa}$ in annealed limit laws of $T_n$. 

Slowdowns - Large Deviations

**Theorem (Gantert & Zeitouni ’98)**

**Case I:** Positive and negative drifts. Let \( v \in (0, v_P) \). Then, \( P_\omega(X_n < nv) \approx e^{-n^{1-1/\kappa}} \), in that for every \( \delta > 0 \),

\[
\limsup_{n \to \infty} \frac{\log P_\omega(X_n < nv)}{n^{1-1/\kappa} + \delta} = -\infty
\]

\[
\liminf_{n \to \infty} \frac{\log P_\omega(X_n < nv)}{n^{1-1/\kappa} - \delta} = 0
\]
**Theorem (Gantert & Zeitouni ’98)**

**Case I:** Positive and negative drifts. Let \( v \in (0, v_P) \). Then,
\[
P_\omega(X_n < nv) \approx e^{-n^{1-1/\kappa}},
\]
in that for every \( \delta > 0 \),
\[
\limsup_{n \to \infty} \frac{\log P_\omega(X_n < nv)}{n^{1-1/\kappa+\delta}} = -\infty
\]
\[
\liminf_{n \to \infty} \frac{\log P_\omega(X_n < nv)}{n^{1-1/\kappa-\delta}} = 0
\]

**Explanation:**

- Deepest trap in \([0, nv]\) has depth \( \approx \frac{1}{\kappa} \log n \)
- Cost to stay in trap \( \approx P(\text{Exp}(n^{-1/\kappa}) > n) = e^{-n^{1-1/\kappa}} \).
Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \wedge 1$. Then, $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)} + o(1)}$, where

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\ \frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \end{cases}$$
Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \wedge 1$. Then, $P(\omega) X_n < n^{\gamma} ) = e^{-n^{\beta(\gamma) + o(1)}}$, where

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa + 1} \\ \frac{\kappa}{\kappa + 1} & \gamma \leq \frac{\kappa}{\kappa + 1} \end{cases}$$

Strategy 1: Stay in deepest trap in $[0, n^{\gamma}]$. 
Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov (’09))

Let $0 < \gamma < \kappa \wedge 1$. Then, $P_\omega(X_n < n^{\gamma}) = e^{-n^{\beta(\gamma)+o(1)}}$, where

$$\beta(\gamma) = \begin{cases} 1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\ \frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \end{cases}$$

Strategy 1: Stay in deepest trap in $[0, n^{\gamma}]$.
- Trap depth is $\approx \frac{1}{\kappa} \log n^{\gamma} = \frac{\gamma}{\kappa} \log n$
- Cost to stay in trap $\approx P(\text{Exp}(n^{-\gamma/\kappa}) > n) = e^{-n^{1-\gamma/\kappa}}$. 
Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov (’09))

Let $0 < \gamma < \kappa \land 1$. Then, $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$, where

$$\beta(\gamma) = \begin{cases} 
1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\
\frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1}
\end{cases}$$

Strategy 1: Stay in deepest trap in $[0, n^\gamma]$.
- Trap depth is $\approx \frac{1}{\kappa} \log n^\gamma = \frac{\gamma}{\kappa} \log n$
- Cost to stay in trap $\approx P(Exp(n^{-\gamma/\kappa}) > n) = e^{-n^{1-\gamma/\kappa}}$.

Strategy 2: Look for a deeper trap farther out and then backtrack.
Theorem (Fribergh, Gantert, & Popov (’09))

Let $0 < \gamma < \kappa \land 1$. Then, $P_{\omega}(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$, where

$$\beta(\gamma) = \begin{cases} 
1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1} \\
\frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1}
\end{cases}$$

---

**Strategy 1:** Stay in deepest trap in $[0, n^\gamma]$.  
- Trap depth is $\approx \frac{1}{\kappa} \log n^\gamma = \frac{\gamma}{\kappa} \log n$ 
- Cost to stay in trap $\approx P(Exp(n^{-\gamma/\kappa}) > n) = e^{-n^{1-\gamma/\kappa}}$.

**Strategy 2:** Look for a deeper trap farther out and then backtrack.  
- Cost to trap in $[0, n^\beta]$ is $\approx e^{-n^{1-\beta/\kappa}}$.
- Cost to backtrack $\approx e^{-n^\beta}$.
Slowdowns - Moderate Deviations

Theorem (Fribergh, Gantert, & Popov ('09))

Let $0 < \gamma < \kappa \land 1$. Then, $P_\omega(X_n < n^\gamma) = e^{-n^{\beta(\gamma)+o(1)}}$, where

$$\beta(\gamma) = \begin{cases} 
\frac{\kappa}{\kappa+1} & \gamma \leq \frac{\kappa}{\kappa+1} \\
1 - \frac{\gamma}{\kappa} & \gamma > \frac{\kappa}{\kappa+1}
\end{cases}$$

**Strategy 1:** Stay in deepest trap in $[0, n^\gamma]$.
- Trap depth is $\approx \frac{1}{\kappa} \log n^\gamma = \frac{\gamma}{\kappa} \log n$
- Cost to stay in trap $\approx P(Exp(n^{-\gamma/\kappa}) > n) = e^{-n^{1-\gamma/\kappa}}$.

**Strategy 2:** Look for a deeper trap farther out and then backtrack.
- Cost to trap in $[0, n^\beta]$ is $\approx e^{-n^{1-\beta/\kappa}}$.
- Cost to backtrack $\approx e^{-n^\beta}$.
- Balance costs: $\beta = 1 - \beta/\kappa$ when $\kappa = \frac{\kappa}{\kappa+1}$.
Proof: Case I

\[ P(X_{2n} = 0) = e^{-n^{\kappa}/(\kappa+1)+o(1)}. \]

**Upper bound:** \( \beta > \frac{\kappa}{\kappa+1} \)

\[
P_\omega \left( \max_{k \leq 2n} |X_k| > n^\beta \mid X_{2n} = 0 \right) \leq \frac{P_\omega(T_{-n^\beta} < \infty)}{P_\omega(X_{2n} = 0)} \leq \frac{e^{-n^\beta}}{e^{-n^{\kappa}/(\kappa+1)}}.
\]

**Lower bound:** \( \beta < \frac{\kappa}{\kappa+1} \)

\[
P_\omega \left( \max_{k \leq 2n} |X_k| \leq n^\beta \mid X_{2n} = 0 \right) \leq \frac{P_\omega(\max_{k \leq 2n} |X_k| \leq n^\beta)}{P_\omega(X_{2n} = 0)} \leq \frac{e^{-n^{1-\beta/\kappa}+o(1)}}{e^{-n^{\kappa}/(\kappa+1)}}.
\]

Cornell University
Jonathon Peterson
2/25/2010
Case II: Positive and zero drift. \( \alpha = P(\omega_0 = \frac{1}{2}) \). Let \( \nu \in (0, \nu_P) \). Then,

\[
P_\omega(X_n < n\nu) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{\pi \log \alpha}{8} (1 - \frac{\nu}{\nu_P}) + o(1) \right) \right\}
\]
Theorem (Povel & Pisztora ('99))

**Case II:** Positive and zero drift. \( \alpha = P(\omega_0 = \frac{1}{2}) \). Let \( v \in (0, v_P) \). Then,

\[
P_\omega(X_n < nv) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{|\pi \log \alpha|^2}{8} \left( 1 - \frac{v}{v_P} \right) + o(1) \right) \right\}
\]

Explanation:
- Travel at typical speed for time \( \frac{v}{v_P} n \).
- Stay in long fair stretch for time \( (1 - \frac{v}{v_P})n \).
Theorem (Povel & Pisztora ('99))

**Case II:** Positive and zero drift. \( \alpha = P(\omega_0 = \frac{1}{2}) \). Let \( v \in (0, v_P) \). Then,

\[
P_\omega(X_n < nv) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{\pi}{8} \frac{|\log \alpha|^2}{(1 - \frac{v}{v_P})} + o(1) \right) \right\}
\]

Explanation:
- Travel at typical speed for time \( \frac{v}{v_P} n \).
- Stay in long fair stretch for time \( (1 - \frac{v}{v_P})n \).
- Longest fair stretch \( \sim \frac{1}{|\log \alpha|} \log n \).
**Case II:** Positive and zero drift. $\alpha = P(\omega_0 = \frac{1}{2})$. Let $v \in (0, v_P)$. Then,

$$P_\omega(X_n < nv) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{\pi \log \alpha}{8} \left( 1 - \frac{v}{v_P} \right) + o(1) \right) \right\}$$

Explanation:
- Travel at typical speed for time $\frac{v}{v_P} n$.
- Stay in long fair stretch for time $(1 - \frac{v}{v_P}) n$.
- Longest fair stretch $\sim \frac{1}{|\log \alpha|} \log n$.
- Small deviations for simple random walk: $r(N) = o(\sqrt{N})$

$$P \left( \max_{k \leq N} |S_k| \leq r(N) \right) \approx \exp \left\{ -\frac{\pi^2}{8} \frac{N}{r(N)^2} \right\}.$$
Proof: Case II

\[ P(X_{2n} = 0) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{\pi \log \alpha}{4} + o(1) \right) \right\} . \]
Proof: Case II

\[ P(X_{2n} = 0) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{|\pi \log \alpha|^2}{4} + o(1) \right) \right\}. \]

**Upper bound:** Backtracking \( n/(\log n)^{2-\varepsilon} \) is too costly

\[ P(T_{-n/(\log n)^{2-\varepsilon}} < \infty) \leq e^{-n/(\log n)^{2-\varepsilon}} \]
Proof: Case II

\[ P(X_{2n} = 0) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{\pi \log \alpha}{4} + o(1) \right) \right\} . \]

**Upper bound:** Backtracking \( n/(\log n)^{2-\varepsilon} \) is too costly

\[ P(T_{-n/(\log n)^{2-\varepsilon}} < \infty) \leq e^{-n/(\log n)^{2-\varepsilon}} \]

**Lower bound:** Confinement in \([-n^\gamma, n^\gamma]\)

\[ P \left( \max_{k \leq 2n} |X_k| \leq n^\gamma \right) = \exp \left\{ -\frac{n}{(\log n)^2} \left( \frac{\pi \log \alpha}{4\gamma^2} + o(1) \right) \right\} \]

Longest fair stretch in \([-n^\gamma, n^\gamma]\) is \(~ \frac{\gamma}{|\log \alpha|} \log n.\)
Case III - Positive Drifts

Change the environment

\[ \tilde{\omega}_x = \frac{\rho_{\text{max}}}{\rho_x + \rho_{\text{max}}} \cdot \]

\( \tilde{\omega} \) has positive and zero drift.

**Proposition (Gantert & P. ('09))**

Let \( B_n = \text{number of visits to sites with } \omega_x \neq \omega_{\text{min}} \text{ in first } 2n \text{ steps}. \) Then, there exists a \( c < 1 \) such that for any \( A \subset \{ X_{2n} = 0 \} \),

\[ P_{\omega}(A) \approx e^{-2nI(0)} E_{\tilde{\omega}}[c^{B_n}1_{\{A\}}] \]
Case III - Positive Drifts

Change the environment

\[ \tilde{\omega}_x = \frac{\rho_{\text{max}}}{\rho_x + \rho_{\text{max}}} . \]

\( \tilde{\omega} \) has positive and zero drift.

**Proposition (Gantert & P. ('09))**

Let \( B_n = \text{number of visits to sites with } \omega_x \neq \omega_{\text{min}} \text{ in first } 2n \text{ steps} \). Then, there exists a \( c < 1 \) such that for any \( A \subset \{ X_{2n} = 0 \} \),

\[ P_{\omega}(A) \approx e^{-2nl(0)} E_{\tilde{\omega}}[c^{B_n} 1_{\{A\}}] \]

Ignoring \( B_n \) we obtain

\[ P_{\omega}(X_{2n} = 0) \leq \exp \left\{ -2nl(0) - \frac{n}{(\log n)^2} \frac{\pi \log \alpha}{2} \right\} \]
Proposition (Gantert & P. (’09))

\[
P_{\omega}(\max_{k\leq 2n} |X_k| \leq n^\gamma, X_{2n} = 0) = \exp \left\{ -l(0)n - \frac{n}{(\log n)^2} \left( \frac{\pi \log \alpha}{\gamma^2} + o(1) \right) \right\}
\]
Confinement Probabilities - Case III

Proposition (Gantert & P. (’09))

\[
P_\omega(\max_{k \leq 2n} |X_k| \leq n^\gamma, X_{2n} = 0) = \exp \left\{ -l(0)n - \frac{n}{(\log n)^2} \left( \frac{|\pi \log \alpha|^2}{\gamma^2} + o(1) \right) \right\}
\]

Explanation:
- Convert to \( \tilde{\omega} \).
- Longest fair stretch in \([-n^\gamma, n^\gamma]\) is \( \sim \frac{\gamma}{|\log \alpha|} \log n \).
Confinement Probabilities - Case III

Proposition (Gantert & P. ('09))

\[
P_\omega(\max_{k \leq 2n} |X_k| \leq n^\gamma, X_{2n} = 0)
\]

\[
= \exp \left\{ -I(0)n - \frac{n}{(\log n)^2} \left( \frac{\pi \log \alpha}{\gamma^2} + o(1) \right) \right\}
\]

Explanation:
- Convert to \( \tilde{\omega} \).
- Longest fair stretch in \([-n^\gamma, n^\gamma]\) is \( \sim \frac{\gamma}{|\log \alpha|} \log n \).
- Restrict to \( B_n \leq \frac{n}{(\log n)^{2-\varepsilon}} \): stay strictly inside fair stretch.