RESEARCH STATEMENT

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My research interests are motivated by problems in quantum mechanics, especially problems in quantization. The analysis is usually done over an infinite dimensional space, where there exists a natural probability measure associated to the space. Stochastic analysis is usually used to understand such spaces. Currently I am working on path space on a manifold.

Path Integrals

It is standard folklore in the physics literature that a classical mechanical system may be "quantized" using Feynman path integrals. More explicitly, suppose that (M,g) is a Riemannian manifold, ∇ is the Levi-Civita covariant derivative, and $V : M \to \mathbb{R}$ is a potential. Then the informal path integral description of "the" quantum mechanical Hamiltonian associated to the classical mechanical system satisfying Newton's equations of motion,

$$\frac{\nabla^2}{dt^2}\sigma(t) = -\mathrm{grad}\ V(\sigma(t)),$$

is given by the heuristic expression

$$(e^{-T\hat{H}}f)(o) := \text{“}\frac{1}{Z_T}\int_{H_T(M)} e^{-\frac{1}{2}E_T(\sigma)-\int_0^T V(\sigma(r))\ dr} f(\sigma(T))\ \mathcal{D}\sigma.\text{”} \tag{1}$$

In this formula, $H_T(M)$ is the space of finite energy paths, $\sigma : [0,T] \to M$, with $\sigma(0) = o \in M$,

$$E_T(\sigma) := \int_0^T g(\sigma'(s), \sigma'(s))\ ds$$

is the energy of the path σ, Z_T is a certain normalization constant and $\mathcal{D}\sigma$ is some sort of " Lebesque " type measure. Unfortunately, the right hand side of the expression is ambiguous. $H_T(M)$ is an infinite dimensional space for which $\mathcal{D}\sigma$ does not exist and the normalization constant Z_T is typically ∞.

My dissertation work involves approximating paths in $H_T(M)$ by piecewise geodesic paths and then passing to the limit to make rigorous sense of the right side of Equation (1). In order to describe the result more precisely we will need the following notation.

Fix $T > 0$. The Wiener space $W_T(M)$ is the space of continuous paths, $\sigma : [0,T] \to M$, with $\sigma(0) = o$ equipped with Wiener measure ν_T on $W_T(M)$. It is well known that the measure ν_T is concentrated on continuous but nowhere differentiable paths. The finite energy path space $H_T(M)$, is a Hilbert manifold. The tangent space $T_\sigma H_T(M)$ to $H_T(M)$ at σ maybe identified with the space of absolutely continuous vector fields X along σ, with one derivative in $L^2(dt)$. Let $G^1(\cdot,\cdot)$ be the metric on $TH_T(M)$ defined by

$$G^1(X,X) = \int_0^T g\Big(\frac{\nabla X(s)}{ds}, \frac{\nabla X(s)}{ds}\Big)\ ds,$$

where ∇ is the Levi Civita covariant derivative.

Let $\mathcal{P}_{n,T} = \{\frac{kT}{n} \mid k = 0,1,\ldots,n\}$, a partition of $[0,T]$ and define $H_{\mathcal{P}_{n,T}}(M)$ to be a set of piecewise geodesics paths in $H_T(M)$ which changes directions only at the partition points of $\mathcal{P}_{n,T}$. It can be shown that $H_{\mathcal{P}_{n,T}}(M)$ is a finite dimensional submanifold of $H_T(M)$ which is diffeomorphic to $(T_oM)^n$. As a submanifold of $H_T(M)$, $H_{\mathcal{P}_{n,T}}(M)$ inherits the metric G^1 from $H_T(M)$, which determines a Riemannian volume measure $Vol^1_{\mathcal{P}_{n,T}}$ on $H_{\mathcal{P}_{n,T}}(M)$.

For each partition $\mathcal{P}_{n,T}$, let $\nu^1_{\mathcal{P}_{n,T}} = \frac{1}{Z^1_{\mathcal{P}_{n,T}}} e^{-\frac{1}{2}E_T} Vol^1_{\mathcal{P}_{n,T}}$ where E_T is the energy of the path σ and $Z^1_{\mathcal{P}_{n,T}}$ is a normalization constant.

Theorem 1 (Lim) *Suppose that M is a d – dimensional compact manifold Riemannian manifold with positive sectional curvature bounded above by $\frac{1}{2dT}$, then for any continuous function f on $W_T(M)$,*

$$\lim_{n\to\infty} \int_{H_{\mathcal{P}_{n,T}}(M)} f(\sigma)\ d\nu^1_{\mathcal{P}_{n,T}}(\sigma) = \int_{W_T(M)} f(\sigma(T)) e^{-\frac{1}{6}\int_0^T S(\sigma(r))\ dr} \sqrt{\det\left(I + \frac{1}{12}K_\sigma\right)}\ d\nu_T(\sigma), \qquad (2)$$

where S is the scalar curvature of the manifold. K_σ is an integral operator defined by

$$(K_\sigma g)(s) \quad = \quad \int_0^T (s \wedge t)\Gamma_{\sigma(t)} g(t)\ dt,$$

where g is a $\mathbb{R}^d$-valued function and $\Gamma_{\sigma(t)}(\cdot)$ is a 2-tensor quadratic in R.

The main strategy of the proof is to write $\int_{H_{\mathcal{P}_{n,T}}(M)} f(\sigma)\ d\nu^1_{\mathcal{P}_{n,T}}(\sigma)$ as an integral over the fixed probability space; $(W_T(\mathbb{R}^d), \mu_T)$ where μ_T is standard Wiener measure. Using the " piecewise geodesic approximation to the stochastic development map ", $\tilde{\phi}_n : W_T(\mathbb{R}^d) \to W_T(M)$, one shows there exists a sequence of densities $\rho_{n,T}(b)$ such that

$$\int_{H_{\mathcal{P}_{n,T}}(M)} f(\sigma)\ d\nu^1_{\mathcal{P}_{n,T}}(\sigma) = \int_{W_T(\mathbb{R}^d)} f(\tilde{\phi}_n(b))\rho_{n,T}(b)\ d\mu_T(b)$$

The density $\rho_{n,T}$ is a function which is given by square root of the determinant of a certain $nd \times nd$ matrix. Because of the complexity of this matrix, it is a challenge to show that $\rho_{n,T}$ is uniformly integrable. The conditions on the sectional curvature of the manifold M were imposed so as to obtain uniform integrability of $\rho_{n,T}$. Using stochastic calculus techniques and perturbation formulas for determinants, one is able to show that $\rho_{n,T}$ converges almost surely to $e^{\rho_T(\tilde{\phi})}$, where $\tilde{\phi}$ is the " stochastic development map ". The formula for $\rho_T(\tilde{\phi})$ can be written down explicitly. It then follows that,

$$\lim_{n\to\infty}\int_{W_T(\mathbb{R}^d)} f(\tilde{\phi}_n(b))\rho_{n,T}(b)\ d\mu_T(b) = \int_{W_T(\mathbb{R}^d)} f(\tilde{\phi}(b))e^{\rho_T(\tilde{\phi}(b))}\ d\mu_T(b).$$

Finally to relate back to the integral on $W_T(M)$, we make use of a fact that $\mu_T \circ \tilde{\phi}^{-1} = \nu_T$ and using the " inverse stochastic development map ", $\tilde{\phi}^{-1}$, we obtain Equation (2).

A Non-Standard Geometric Quantization of the Simple Harmonic Oscillator

The setting for a classical phase space is a symplectic manifold. The symplectic 2-form ω gives a volume form which allows one to define $L^2(B)$, where B is a Hermitian line bundle with covariant derivative ∇ whose curvature is $\frac{1}{i\hbar}\omega$. Classical observables H easily become operators $Q(H) := -i\hbar\nabla_{X_H} + M_H$ where X_H is the Hamiltonian vector field associated with H and M_H is the multiplication operator with H. This is called prequantization. Unfortunately, this prequantum Hilbert space $L^2(B)$ is too big. If M is a Kähler manifold with Kähler potential ϕ, then the Quantum Hilbert space $\mathbb{H}$ is defined to be the space of all holomorphic line bundles which are L^2 integrable with respect to the measure $e^{-\phi}d\omega^n$.

Consider a harmonic oscillator in $M = \mathbb{R}^2$, which is always a Kähler manifold. The function being quantized here is the Hamiltonian, $H = \frac{1}{2}p^2 + \frac{1}{2}x^2$ where (x, p) are the coordinates of $\mathbb{R}^2$. In this case, the line bundle is $B = M \times \mathbb{C}$ and $\nabla = d + \frac{1}{2i\hbar}(pdx - xdp)$. Equipped with an almost complex structure

$$J_{std} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

the holomorphic coordinate $z := x - ip$ is compatible with the almost complex structure J_{std}, with corresponding Kähler potential ϕ_{std}. The Quantum Hilbert space $\mathbb{H}_{std}$ is given by all holomorphic functions which are L^2 integrable when integrated against the measure $e^{-\phi_{std}}dxdp$. The quantized operator $Q_{std}(H)$, in holomorphic coordinate is given by $z\frac{d}{dz}$, which preserves $\mathbb{H}_{std}$. This is one form of the standard quantization of the simple harmonic oscillator.

However, J_{std} is not the only almost complex structure one can assign to $\mathbb{R}^2$. Given another almost complex structure J, there exists holomorphic coordinate $w(x, p)$ compatible with J, with corresponding Kähler potential ϕ. The exact same procedure is carried out in the quantization of H, in the new holomorphic coordinate w. The Quantum Hilbert space $\mathbb{H}$ is given by all holomorphic functions in w which are L^2 integrable when integrated against the measure $e^{-\phi}dxdp$. If $L_{X_H}J = 0$ where L denotes the Lie derivative, then the quantized operator $Q(H)$ in this case is a holomorphic vector field which preserves $\mathbb{H}$.

Theorem 2 (Lim) *Let (r,θ) be polar coordinates and a, b be smooth radial functions such that $\lim_{r\to 0} a(r) = 0$ and $\lim_{r\to 0} b(r) = 1$. If $J_{non\text{-}std}$ is of the form*

$$J_{non\text{-}std} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a(r) & b(r) \\ -\frac{1+a^2(r)}{b(r)} & -a(r) \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

then

1. *$L_{X_H} J_{non\text{-}std} = 0$ so that quantization of H, $Q_{non\text{-}std}(H)$ is unambiguously defined.*

2. *However, the quantized operator $Q_{non\text{-}std}(H)$ is **not** unitarily equivalent to $Q_{std}(H)$.*

Current and Future Research

Define an operator $\mathcal{L}$ by

$$(\mathcal{L}f)(o) = \int_{W_T(M)} f(\sigma(T)) e^{-\frac{1}{6}\int_0^T S(\sigma(r))\ dr} \sqrt{\det\left(I + \frac{1}{12}K_\sigma\right)}\, d\nu_T(\sigma).$$

I am currently working on a proof to show that $\mathcal{L} = e^{-T\hat{H}}$ for some differential operator $\hat{H}$. Using the path integral prescription, $\hat{H}$ will be the quantized hamiltonian.

Theorem 1 was stated for compact manifolds with positive curvature, with an upper bound on the sectional curvature. I plan to investigate if the upper bound on the curvature is necessary. Another possible extension of Theorem 1 would be to remove the hypothesis of positive sectional curvature. I would also like to obtain a similar result for pinned Wiener measure.

A long term goal will be to make sense of Equation (1), where instead of a path, we have an embedded surface in a manifold M.

In the non standard geometric quantization of the harmonic oscillator, I plan on investigating the 'size' of the Quantum Hilbert space $\mathbb{H}$. More precisely, what is the dimension of $\mathbb{H}$ and the domain of $Q_{non-std}(H)$?

With my knowledge on stochastic calculus and differential geometry, I am also interested in working on other problems in these areas. In particular, stochastic models have many applications in areas such as finance, which I would like to explore in the future.