Homework (due on Tuesday, 02/02/10)

Problem 1.

Learn about Snell’s law (on internet or from textbooks) & re-derive it using the following example:

Suppose the speed of motion is $c_1$ in the left half-plane and $c_2$ in the right half-plane. You are trying to find the optimal (quickest) path from point A with coordinates (-1, 0) to point C with coordinates (1, 1). That optimal path passes through some point B with coordinates (0, y), where y is an unknown number from [0,1].

Suppose y is in (0,1). Use calculus to write down the conditions which have to be true if the path ABC is truly optimal.

Suppose A’ has coordinates (-1, y) and C’ has coordinates (1, y). Denote the angle A’BA as alpha and the angle C’BC as gamma. Use the above to show that

$$\frac{\sin(\alpha)}{\sin(\gamma)} = \frac{c_1}{c_2}.$$

Problem 2.

Consider two paths from Ithaca (I) to Toronto (T):

Path I:
the shortest path from I to T as returned by MapQuest or similar website;
(this will lead you along the western side of Cayuga Lake)

Path II:
the shortest path from I to Syracuse (S) + the shortest path from S to T;
(this will lead you through a stretch of 81N).

Path I is clearly faster, but imagine that somewhere on route NY-96 there might be a 10-mile-stretch covered with ice/snow, through which you cannot drive faster than 5 mph.
Suppose the probability of this disaster happening is (1/2) and you start out knowing only the place X on NY-96 starting from which this deterioration of driving conditions might be happening.

It is clear that there are 3 different strategies available to you:

Strategy I: try Path I first; if the snow/ice is encountered at X, go back to Ithaca & revert to Path II.
Strategy II: take Path II right away, without wasting time to check Path I.
Strategy III: always take Path I and follow it through to the end (whether or not there is any snow at X).

Can you find any example of X for which Strategy II is the best?
Can you find an example of X such that Strategy II is the best.
Can you find the north-most position of X, for which Strategy I is the best?

Problem 3.

Consider a gigantic (or infinite) checkerboard composed of small white and black squares with sidelength epsilon. Suppose you can travel with speed 1 through the black squares and with speed 2 through the white squares.
Pick a point A and define a function $T(x)$ for the least-time-needed to travel from A to each x. All level-curves of $T(x)$ will have many “wiggles” of size comparable to (epsilon). If you look at the level-curves which are relatively close to A on a relatively small checkerboard, all you can see is some very complicated (but symmetric) pattern. For example, on an (11 by 11) board with the point A in the center, you will see this:

However, if you consider a much larger board and look at the level-curve $T(x)= (100 \text{ epsilon})$ or $T(x) = (1000 \text{ epsilon})$ or even further, you will see some simple geometric shape centered at A on a large scale though covered (yet again) by many small “wiggles” of size comparable to (epsilon).

Your goal is to determine what that simple geometric shape is.

Hints:
One thing that might be helpful is to think of various directions of motion & to determine the optimal strategy for traveling from A in each direction exploiting the checkerboard pattern to minimize the total time.

E.g., suppose $b$ is a unit-length vector specifying some direction and $r$ is the distance that we need to travel in that direction. Let $x(b, r) = A + r \cdot b$ and suppose, as before, that $T(x(b, r))$ is the minimum time needed to travel from A to that point. Then we can define the “homogenized speed of travel” in the direction $b$ by taking the limit of $[ r / T(x(b, r)) ]$ as $r$ tends to infinity.

You should start by finding all vectors $b$ for which this speed is maximal. For all the other directions, can you somehow use the already found “fast directions” to build an efficient motion strategy? What speed of motion will that strategy yield? Use this info to recover the shape of far-away-level-curves of $T(x)$. 