The Problem of Moduli for Complex Analytic Manifolds

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In section 3, we will constantly use the idea — to be made familiar — of a Banach manifold and the related idea of a $\mathbb{C}$-analytic Banach space.

If $U$ is an open set in a Banach space $E$, and $\Theta$ a $\mathbb{C}$-analytic transformation of $U$ to another Banach space $F$, the $\mathbb{C}$-analytic functions $h$ on an open subset of $U$ with values in $\mathbb{C}$ have the form $(\Theta(u), f(u))$, where $f$ is a $\mathbb{C}$-analytic transformation with values in the dual $F'$ of $F$, forming an ideal $\mathcal{J}$ of the sheaf $\mathcal{O}(U)$. We assign to $X = \Theta^{-1}(0)$ the sheaf $\mathcal{O}/\mathcal{J}$. For the purpose of exposition, it suffices to define the $\mathbb{C}$-analytic Banach spaces to be locally isomorphic to a model of this type.

1 The Manifold of Complex Structures on a Vectorspace

Let $E$ be a vectorspace of dimension $2n$ over $\mathbb{R}$. For all complex structures $\phi$ on $E$, there is a unique $\mathbb{C}$-linear transformation from $\mathbb{C} \otimes E$ to $(E, \phi)$ [prolonging the identity?]; its kernel $K_\phi$ is a complex subspace of $\mathbb{C} \otimes E$. We therefore obtain a bijection from the space $\Phi(E)$ of complex structures on $E$ to the space of complex subspaces $K$ of $\mathbb{C} \otimes E$ such that $\mathbb{C} \otimes E = K \otimes \bar{K}$. When $\Phi(E)$ is identified with an open subset of the Grassmannian of $\mathbb{C} \otimes E$, it then inherits the structure of a complex analytic manifold.

Let $\phi_0$ be a complex structure on $E$; set $K_0 = K_{\phi_0}$. For all complex structures $\phi$ sufficiently close to $\phi_0$, $K_\phi$ is the graph of a $\mathbb{C}$-linear transformation $u_\phi : K_0 \rightarrow \bar{K}_0$. The projection $p''$ (resp. $p'$) from $E$ to $K_0$ (resp. $K_\phi$) is an isomorphism (resp. anti-isomorphism) for $\phi_0$[?]. To $\phi$ there corresponds a $\mathbb{C}$-linear transformation $\omega_\phi = p''^{-1} \circ u_\phi \circ p'$ from $(E, \phi_0)$ to itself, and the map $\phi \mapsto \omega_\phi$ is a chart on $\Phi(E)$ defined near $\phi_0$. 

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This transformation identifies \((E, \phi_0)\) with \((E, \phi)\) and is \(\mathbb{R}\)-linear, so it is of the form \(f' + f''\), where \(f'\) is \(\mathbb{C}\)-linear and \(f''\) is \(\mathbb{C}\)-antilinear. We then have \(\omega_\phi = f^{-1} \circ f''\).

## 2 The Relative Frobenius Complex

Let \(V_0 = (V, \phi_0)\) be a compact complex analytic manifold, and let \(V\) be the \(\mathbb{R}\)-analytic sub-bundle [sous-jacent] corresponding to the complex structure \(\phi_0\). By considering for each point \(x \in V\) the chart of \(\Phi(T_X(V))\) defined for \(\phi_0(x)\), we associate a unique almost-complex \(\mathbb{R}\)-analytic structure \(\phi\) on \(V\) close enough to \(\phi_0\) in the \(C^1\) sense, an \(\mathbb{R}\)-analytic morphism which is \(\mathbb{C}\)-antilinear from the tangent fibers \(T(V_0)\) to themselves, i.e. a form \(\omega_\phi\) of type \((0, 1)\) with values in \(T(V_0)\). For \(\phi\) to be integrable, it suffices for \(\omega = \omega_\phi\) to satisfy \(d''\omega - [\omega, \omega] = 0[1]\). In this case \(\phi\) is a sub-bundle [sous-jacent?] with a unique \(\mathbb{C}\)-analytic structure (see for example [3], p. 36).

Consider the space \(\Phi(V)\) of pairs \((x, \phi)\), where \(x \in V\) and \(\phi \in \Phi(T_X(V))\); this is a \(\mathbb{C}\)-analytic bundle over \(V\) (i.e., an \(\mathbb{R}\)-analytic manifold given by the submersion of \(V\) and a \(\mathbb{C}\)-analytic structure on the fibers). If \(S\) is a \(\mathbb{C}\)-analytic space, \(S \times V\) is a \(\mathbb{C}\)-analytic bundle over \(V\). We then define an \(\mathbb{R}\)-analytic almost-complex structure on \(V\), parameterized by \(S\) to be an \(\mathbb{R}\)-analytic morphism, \(\mathbb{C}\)-analytic on the fibers, of \(\mathbb{C}\)-analytic bundles over \(V\), \(S \times V \longrightarrow \Phi(V)\). Likewise, we define the forms of type \((p, q)\) with values in \(T(V_0)\) relative to \(S\). We then define the operators \(d''\) and \([,\] \) on these forms, again relative to the parameter \(S\). An almost-complex structure \(\phi\) on \(V\) parameterized by \(S\) will be called integrable if the form \(\omega = \omega_\phi\) parameterized by \(S\) satisfies the equation \(d''\omega - [\omega, \omega] = 0\).

**PROPOSITION 1.** Let \(\phi\) be an integrable \(\mathbb{R}\)-analytic almost-complex structure on \(V\) parameterized by \(S\). Then \(\phi\) has unique \(\mathbb{C}\)-analytic sub-bundle [sous-jacent] structure in \(S \times V\), which we also will call \(\phi\). On setting \(X = (S \times V, \phi)\), the projection \(X \longrightarrow S\) is a smooth map, and for any \(x \in X\), \(S \times \{x\}\) is a \(\mathbb{C}\)-analytic subspace of \(X\).

The proof is analogous to that of [3].

## 3 Manifolds of Transformations

Suppose \(M\) is a compact manifold of class \(C^r\), and \(V\) a \(\mathbb{C}\)-analytic manifold. The space \(C^r(M; V)\) is equipped with the structure of a \(\mathbb{C}\)-analytic Banach manifold, and

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1. The lack of difficulty comes from the assumption that \(\phi\) is \(\mathbb{R}\)-analytic. There is a far more delicate introduction of parameters in Nirenberg’s proof for the \(C^\infty\) case.
$T^r C^r(M; V)$ is identified with the Banach space of $C^r$-sections in the fibers $f^* T(V)$ on $M$.

More generally, suppose $S$ is a $C$-analytic space and $X$ a $C$-analytic bundle over $S$. The space $C^r(M; X)$ of $C^r$ maps from $M$ to the fibers of $X$ is a trivial $C$-analytic Banach bundle over $S$; that is, at any point of $M$ there is a neighborhood on which $f : M \to X_S$ [induces?] is an $S$-isomorphism with the product of the a neighborhood of $s \in S$ with an open set in a Banach space.

Likewise, if $E$ is a $C$-analytic bundle over $M$, the space of $C^r$-sections of $E$ is given the structure of a $C$-analytic Banach manifold. In particular, if $M$ is a manifold of class $C^{r+1}$, the set $\Phi^r(M)$ of almost-complex structure of class $C^r$ on $M$ is a $C$-analytic Banach manifold.

With these notations we have defined the set $\text{Diff}^{r+1}(M; X)$ of $C^{r+1}$ diffeomorphisms from $M$ to the fiber of $X$ is an open set in $\text{Diff}^r(M; X)$, and we have a $C$-analytic map of $\text{Diff}^{r+1}(M; X)$ to $\Phi^r(M)$ which from $f : M \to X_S$ associates $f^*(\phi_S)$, where $\phi_S$ is the complex structure on $X_S$.

The same considerations do not entirely apply to the case $r = k + \alpha \ [1]$. One could instead use the maps of class $H^s$ if $s$ is large enough.

4 Local Study of $\Phi^r(V)$

Suppose, as in section 2, $V_0 = (V, \phi_0)$ is a compact $C$-analytic manifold. We define $\Omega^{p,q}$ to be the Banach space of $C^r$ forms of type $(p, q)$ on $V_0$ with values in the tangent fibers $T(V_0)$. The chart defined by $\phi_0$ takes a neighborhood of $\phi_0$ to $\Phi^r(V)$ and a neighborhood of 0 to $\Omega^{0,1}$. [2]

The $C$-analytic Banach space $\Psi^r(V)$ of integrable structures is defined on a chart by the equation $\Theta(\omega) = 0$, where $\Theta : r\Omega^{0,1} \to r^{-1} \Omega^{0,2}$ is the analytic map defined by $\Theta(\omega) = d''\omega - [\omega, \omega]$. The tangent map of $\Theta$ is $d'' = d''$. If $r > 1$ is non-integral[?], which we will assume from now on,

\[ d' : r\Omega^{0,0} \to r\Omega^{0,1}, \quad d'' : r\Omega^{0,1} \to r^{-1}\Omega^{0,2} \]

are homomorphisms with closed image. Let us give $V_0$ an $\mathbb{R}$-analytic hermitian structure, and define $\delta'_1$ and $\delta'_2$ to be the adjoints of $\delta'_0$ and $\delta'_1$ relative to this metric. We then have

\[ r\Omega^{0,1} = \text{Im} \ d''_0 \oplus \ker \delta'_1 \quad \text{and} \quad r\Omega^{0,2} = \text{Im} \ d''_1 \oplus \ker \delta'_2 = \{ \text{dothis or dothis?} \} \]

Suppose $\Sigma = \Theta^{-1}(\ker \delta'_2)$. The implicit function theorem then shows that, in a neighborhood fo $O$, $\Sigma$ is a $C$-analytic Banach submanifold of $r\Omega^{0,1}$ containing $\Phi^r(V)$, and $T_0(\Sigma) = \ker d''_1$. 


Suppose \( H = \Sigma \cap \text{Ker } \delta'_1 \); close to 0, \( H \) is a \( \mathbb{C} \)-analytic submanifold of finite dimension in \( \Sigma \), and \( T_0(H) \) is the vector space of harmonic \((0, 1)\)-forms with values in \( T(V_0) \), which may be identified with \( H^1(V_0; \Theta) \), when \( \Theta \) is the sheaf of holomorphic vector fields on \( V_0 \). \( H \) can be defined by the equation

\[
\delta'(d'' \omega - [\omega, \omega]) + d'' \delta' \omega = 0.
\]

By the theory of elliptic equations, any form \( \omega \in H \) is \( \mathbb{R} \)-analytic, and the injection of \( H \) into \( \Omega^{0,1}(V) \) defines on \( V \) the almost-complex structure parameterized by \( H \).

Now consider the subspace \( S \subset H \) defined by \( \Theta(\omega) = 0 \). Its Zariski tangent space at 0 is \( T_0(H) \). The injection of \( S \) into \( H \) defines on \( V \) an integrable \( \mathbb{R} \)-analytic almost-complex structure parameterized by \( S \). We note that \( X \) is the \( \mathbb{C} \)-analytic space obtained from \( S \times V \) by providing the structure we have just defined.

5 Kuranishi’s Theorem

The \( \mathbb{C} \)-analytic space \( X \), smoothly and properly embedded in the marked space \((S, s_0) = 0\), along with the identification \( i : V_0 \longrightarrow X_0 \), enjoys the following semi-universal property:

THEOREM: For all \( \mathbb{C} \)-analytic marked spaces \((S', s'_0)\), all \( \mathbb{C} \)-analytic spaces \( X \) smoothly and properly embedded in \( S' \), and for all isomorphisms \( i' : V_0 \longrightarrow X'_0 \), there exists a neighborhood \( S'' \) of \( s'_0 \) in \( S' \), a map \( f : (S'', s'_0) \longrightarrow (S, s_0) \) (not necessarily unique), and a \( S'' \)-isomorphism \( g : X'|_{S''} \longrightarrow f^*(X) \) such that \( g_{s'_0} = i \circ i'^{-1} \).

Proof: Consider the \( \mathbb{C} \)-analytic smooth subspace \( \mathcal{D} = \text{Diff}^{r+1}(V; X) \) of \( S \) with the basepoint \( e = I_{V_0} \). We have a section \( \sigma : S \longrightarrow \mathcal{D} \) coming from the identity \( S \times V = X \), and a morphism \( \rho : \mathcal{D} \longrightarrow \Phi^r(V) \) such that, for \( f : V \longrightarrow X_S \), \( \rho(f) = f^*(\phi_S) \), where \( \phi_S \) is the complex structure on \( X_S \). Then the composition \( \rho \circ \sigma : S \longrightarrow \Phi^r(V) \) is a canonical injection. The tangent map of \( \rho : \mathcal{D}_0 = \mathcal{D}_{s_0} \longrightarrow \Phi^r(V) \) at \( e \) is \( d'_0 : r+1\Omega^{0,0} \longrightarrow r\Omega^{0,1} \), which is a homomorphism with finite-dimensional kernel.

Let \( \mathcal{E} \) be a \( \mathbb{C} \)-analytic Banach subspace of \( \mathcal{D} \), smoothly embedded in \( S \), containing \( \sigma(S) \), and such that \( r+1\Omega^{0,0} = \text{Ker } d'_0 \oplus T(\mathcal{E}_0) \). The restriction of \( \rho \) to \( \mathcal{E} \) is an immersion of \( \mathcal{E} \) into \( \Sigma \) at \( e \), because the tangent map is an isomorphism. We have \( S \subset \rho(\mathcal{E}) \subset \Phi^r(V) \subset \Sigma \), these inclusions to be seen as analytic maps. By the following lemma we will show that

\[
\rho(\mathcal{E}) = \Phi^r(V) \quad \text{in a neighborhood of } 0. \tag{1}
\]
LEMMA 1. Let $H$ be a neighborhood of 0 in $C^k$, $U$ a neighborhood of 0 in a Banach space, and $\Phi$ a $\mathbb{C}$-analytic subspace of $H \times U$. Suppose $S = (H \times \{0\}) \cap \Phi$. If $\Phi \supset S \times U$, we have $\Phi = S \times U$ in a neighborhood of 0 on $H \times U$.

The lemma is proven in section 6.

For proving the assertion (1) from the lemma, it suffices to trivialize $\mathcal{E}$ in a neighborhood of $e$, and to extend $\rho: \mathcal{E} = S \times U \longrightarrow \Sigma$ to a $\mathbb{C}$-analytic map of $H \times U$ into $\Sigma$, which is then a chart on $\Sigma$.

Now let $(S', s'_0)$ be a $\mathbb{C}$-analytic space with basepoint, $X'$ a $\mathbb{C}$-analytic space bundle over $S'$, and $i': V_0 \longrightarrow X'$ an isomorphism. The analytic space $\mathcal{D}' = \text{Diff}^{r+1}(V, X')$ being smooth[?], there exists a $\mathbb{C}$-analytic section $\sigma': S'' \longrightarrow \mathcal{D}'$ on a neighborhood $S''$ of $s'_0$ in $S'$ such that $\sigma'(s'_0) = i'$. Additionally, we have as before a morphism $\sigma': \mathcal{D}' \longrightarrow \Phi'(V)$ such that, for $f': V \longrightarrow X'_{s'}$, $\rho(f) = f^*(\phi'_{s'})$. Since $\rho: \mathcal{E} \longrightarrow \Phi'(V)$ is a local isomorphism at the point $e = i$ one can, by decreasing $S''$, write $\rho' \circ \sigma'$ in a unique way as $\rho \circ h$, where $h$ is a $\mathbb{C}$-analytic map from $(S'', s'_0)$ to $(\mathcal{E}, e)$.

Composing $h$ with the projection $\mathcal{E} \longrightarrow S$, we obtain a map $f: (S'', s'_0) \longrightarrow (S, s_0)$. To $h$ (resp. to $\sigma'$) there corresponds an $f$-morphism (resp. an $S''$-morphism) $\tilde{h}$ (resp. $\tilde{\sigma}'$) of $S'' \times V$ to $X$ (resp. to $X'$), of class $C''$, partially $\mathbb{C}$-analytic with respect to $S''$. The relation $\rho' \circ \sigma' = \rho \circ \sigma$ shows that the almost-complex structures which $S''$ induces on $V$ by $X$ and $X'$ through $\tilde{h}$ and $\tilde{\sigma}'$ coincide, so that the map $g = \tilde{h} \circ \tilde{\sigma}'^{-1}: X'_{s'} \longrightarrow X$ is $\mathbb{C}$-analytic, which completes the proof of the theorem.

Remark: $f$ depends on the choice of $\mathcal{E}$. If $H^0(V_0; \Theta) = 0$, we must have $\mathcal{E} = \mathcal{D}$ on a neighborhood of $e$; in this case, $f$ and $g$ are unique, and $X \longrightarrow S$ enjoys a universal property.

6 Proof of Lemma 1

If $f$ is a function on $H \times U$ and $u \in U$, we denote by $f_u$ the function $x \mapsto f(u, x)$. For all polycylinders $K \subset H$ of rays $r_1, \ldots, r_k$, we let $B_K$ denote the Banach space of continuous functions on $B_K$ [surely $K$?] which are holomorphic on the interior [4].

Now let $f_1, \ldots, f_p$ be $\mathbb{C}$-analytic functions on a neighborhood of 0 in $H \times U$ belonging to the ideal defined by $\Phi$, and such that $f_0, \ldots, f_p$ generate the ideal $\mathcal{J}$ defined by $S$ in a neighborhood of 0; we define $f_u$ to be the homomorphism of sheaves $\mathcal{O}^p \longrightarrow \mathcal{O}$ defined by $f_u(h_1, \ldots, h_p) = \sum f_u^i h_j$. If $K$ is a privileged neighborhood of 0 in $H$ for $f$, i.e. a polycylinder such that $f_0$ induces a homomorphism $B_K^p \longrightarrow B_K$ having the closed subspace $I_K \subset B_K$ as the image, where $I_K$ is formed of functions $h$ such that the germ at 0 is in $\mathcal{J}$. Since $\Phi \supset S \times U$, we have $f_u(B_K) \subset I_K$ for all $u$ and, since $f_0$ is an epimorphism, $f_u$ will be an epimorphism for $u$ sufficiently close.

\footnote{In the sense of analytic spaces. The lemma is false, if one assumes only inclusion as sets.}
to 0. We can therefore write \( f_0 \) in the form \( \sum g_{ij}^u f_j^u \) and, by utilizing a complement of Ker \( f_0 \), we can choose the \( g_{ij}^u \) to depend analytically on \( u \). This shows that the functions \((x, u) \mapsto f_0^u(x)\), which generate the ideal defining \( S \times U \), also belong to the ideal defining \( \Phi \). Q.E.D.

7 A Description of \( S \)

\( H \) is a submanifold of \( r^1\Omega^{0,1} \), and \( T_0(H) = H^1 = \text{Ker } d''_1 \cap \text{Ker } \delta'_1 \) is identified with \( H^1(V_0; \Theta) \); there exists a neighborhood \( U \) of 0 in \( H^1 \) and a chart \( \eta : U \to H \) such that \( \eta(u) = u + o(\|u\|) \). The subspace \( S \) of \( H \) is defined by \( \Theta(\omega) = 0 \), where \( \Theta \) is the analytic map of \( H \) to \( r^1\Omega^{0,2} \) defined by \( \Theta(\omega) = d''_2 \omega + [\omega, \omega] \).

**PROPOSITION 2.** There exists a \( C \)-analytic map \( \zeta : U \to H^2(V_0; \Theta) \) such that \( \eta \) induces a local isomorphism \( \zeta^{-1}(0) \to S \). We can choose \( \zeta \) to be of the form \( \zeta(u) = [u \cup u] + o(\|u\|^2) \), where \( [u \cup u] \) is the cup product.

**Proof:** We have the equations \( \Theta(\omega) \in \text{Ker } \delta'_2 \) and \( d''(\Theta(\omega)) = 2[\omega, \Theta(\omega)] \) for \( \omega \in H \). Additionally,

\[
r^{-2}\Omega^{0,3} = \text{Im } d''_2 \oplus \text{Ker } \delta'_3.
\]

Now consider the set \( F \) of pairs \((\omega, \alpha) \in H \times \text{Ker } \delta'_2\) such that \( d'' \alpha - 2[\omega, \alpha] \in \text{Ker } \delta'_3\). The implicit function theorem implies that \( F \) is a sub-vectorbundle of \( H \times \text{Ker } \delta'_2\), and the projection \( p : r^{-1}\Omega^{0,2} \to H^2 = \text{Ker } d''_2 \cap \text{Ker } \delta'_2 \) induces a local isomorphism of \( F \) onto \( H \times H^2 \). Since \( \Theta \) defines a section of \( F \), we have

\[
\Theta^{-1}(0) = (p \circ \Theta)^{-1}(0),
\]

and \( \zeta = -p \circ \Theta \circ \eta \) satisfies the first condition.

We have

\[
-p(\Theta(\omega)) = p([\omega, \omega]),
\]

and

\[
\zeta(u) = p([\eta(u), \eta(u)]) - p([u, u]) + o(\|u\|^2).
\]

However, \([u \sim u] = p([u, u])\), which completes the proof.

**References**


   *See also:*