Pathspace Connections and 2-Groups

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January 30, 2006

1 Motivation

Throughout these notes, a fixed local trivialization of all principal bundles is assumed. This allows us to simplify some calculations by dealing with $\mathfrak{g}$-valued $k$-forms and $G$-valued functions directly.

1.1 Fundamental Theorem of Calculus

Let us first recall why electromagnetism is easy:

**Lemma 1.1.** (Poincaré, Gauss, Leibniz, ...) There is an exact sequence of sheaves

\[ 1 \longrightarrow U(1) \xrightarrow{C} \mathcal{C}^{\infty}(X, U(1)) \xrightarrow{d \log} \Omega^{1}_{X}(i\mathbb{R}) \xrightarrow{d} \Omega^{2}_{X}(i\mathbb{R}) \xrightarrow{d} \Omega^{3}_{X}(i\mathbb{R}) \xrightarrow{d} \ldots \]

where $U(1)$ is the sheaf of locally constant functions.

In particular, if we are given a 2-form $F \in \Omega^{2}_{X}(i\mathbb{R})$, we know that $F$ locally comes from a potential $A \in \Omega^{1}$ if and only if $dF = 0$. This means we can easily test which 2-forms could be electromagnetic fields, and finding a potential is no more difficult than finding an antiderivative.

For more general Yang-Mills gauge fields, the group $U(1)$ will be replaced with a nonabelian group $G$. In this case, we at least have

**Lemma 1.2.** (Cartan?) There is an exact sequence of sheaves

\[ 1 \longrightarrow G \xrightarrow{C} \mathcal{C}^{\infty}(X, G) \xrightarrow{\ast \theta} \Omega^{1}_{X}(\mathfrak{g}) \xrightarrow{\text{curv}} \Omega^{2}_{X}(\mathfrak{g}) \]

where $\ast \theta : f \mapsto f^{-1} \cdot df$ and \text{curv} $\omega = df + \omega \wedge \omega$.

Unfortunately, the obvious necessary condition for $F \in \Omega^{2}(\mathfrak{g})$ to satisfy \text{curv} $\omega = F$ for some $\omega \in \Omega^{1}(\mathfrak{g})$ is the Bianchi identity

\[ 0 = d^2 F = dF + \omega \wedge F \]

which explicitly depends on already having the gauge potential $\omega$ on hand.
1.2 From Space to Pathspace

Let \( \tilde{P}X \) denote the space of \([0,1]\)-parametrized, piecewise smooth paths in \( X \). Similarly, let \( P X \) be the space of unparametrized (but oriented) piecewise smooth paths in \( X \).

It is natural to ask what the fundamental theorem of calculus looks like from the perspective of pathspace. If \( \omega \in \Omega^1_X(i\mathbb{R}) \) is given, then we can get a function \( P : P X \to U(1) \) by

\[
P(\gamma) = \exp \left( \int_\gamma \omega \right)
\]

Likewise, a \( k \)-form \( \eta \in \Omega^k_X(i\mathbb{R}) \) gives a \((k-1)\)-form \( \tau(\eta) \in \Omega^{k-1}_{P X}(i\mathbb{R}) \) by the formula

\[
\tau(\eta)(\gamma)(v_1, ..., v_{k-1}) = \int_0^1 \eta(v_1(t), ..., v_{k-1}(t), \gamma'(t)) \, dt
\]

where the \( v_i \) are vectorfields along \( \gamma \) (that is, tangent vectors to \( P X \) at \( \gamma \)). Note that this formula does not depend on the parametrization of \( \gamma \).

The above discussion can be summarized by this suggestive diagram, where \( G = U(1) \), \( \Omega^k \) means \( \Omega^k(g) \), and \( C^\infty_X \) means \( C^\infty(X,G) \):

\[
\begin{array}{ccccccccc}
1 & \rightarrow & G & \rightarrow & C^\infty_X & \xrightarrow{d \log} & \Omega^1_X & \xrightarrow{d} & \Omega^2_X & \xrightarrow{d} & \cdots \\
\downarrow && \downarrow && \downarrow && \exp f && \tau && \tau \\
1 & \rightarrow & G_P & \rightarrow & C^\infty_{P X} & \xrightarrow{d \log} & \Omega^1_{P X} & \xrightarrow{D} & \Omega^2_{P X} & \xrightarrow{D} & \cdots \\
\end{array}
\]

Here, \( G_P \) means “functions on pathspace which only depend on the endpoints”.

**Lemma 1.3.** Everything in this diagram commutes.

**Proof.** First, we need to explain the map \( D : \Omega^k_{P X} \to \Omega^{k+1}_{P X} \). This is essentially the exterior derivative on pathspace, with a corrective factor thrown in for the motion of the endpoints. Explicitly, if \( \omega \in \Omega^k_{P X}(i\mathbb{R}) \) is given and we choose a curve \( \gamma \), then there is some \( k \)-form \( \hat{\omega} \) such that

\[
\omega(\gamma)(v_1, ..., v_k) = \int_0^1 \hat{\omega}(\gamma(t))(v_1(t), ..., v_k(t)) \, dt
\]

Now we define

\[
(D\omega)(\gamma)(v_0, ..., v_k) = (d\omega)(\gamma)(v_0, ..., v_k) + d\hat{\omega}(\gamma(0))(v_0(0), ..., v_k(0)) - d\hat{\omega}(\gamma(1))(v_0(1), ..., v_k(1))
\]

\[1\text{st ad hoc, it is motivated by thinking of the pathspace } \mathcal{P}X \text{ as a sort of smooth category.} \]
Let us just prove that the square

\[
\begin{array}{ccc}
\Omega^1_X & \xrightarrow{d} & \Omega^2_X \\
\exp f & \downarrow & \tau \\
C^{\infty}_{\mathcal{P}X} & \xrightarrow{D\log} & \Omega^1_{\mathcal{P}X}
\end{array}
\]

commutes, the general case being just a messier version of this. Starting with an arbitrary \( \omega \in \Omega^1_X \), write

\[
\omega = \sum_{i=1}^n \omega_i \, dx^i, \quad d\omega = \frac{1}{2} \sum_{i,j=1}^n \left( \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) \, dx^i \wedge dx^j
\]

Now pick a curve \( \gamma \) and a vectorfield \( v = v^i \, \partial/\partial x^i \) along it. Without (much) loss of generality, assume that \( \gamma' \) is linearly dependent on \( \partial/\partial x^1 \). Then we have

\[
\tau(d\omega)_\gamma(v) = \int_0^1 d\omega_{\gamma(t)}(v(t), \gamma'(t)) \, dt
\]

\[
= \sum_{i=1}^n \int_0^1 \left( \frac{\partial \omega_i}{\partial x^1} - \frac{\partial \omega_1}{\partial x^i} \right) \cdot v^i \cdot \gamma' \, dt
\]

\[
= \int_0^1 \partial_i \omega_1 \cdot \gamma' \, dt - \sum_{i=1}^n \int_0^1 \frac{d\omega_i}{dt} \cdot v^i \, dt
\]

\[
= \int_0^1 \partial_i \omega_1 \cdot \gamma' + \omega \left( \frac{dv}{dt} \right) \, dt + \omega_{\gamma(0)}(v(0)) - \omega_{\gamma(1)}(v(1))
\]

On the other hand, we can compute \( d \int \omega \) and add in boundary terms to find \( D \log \exp \int \omega \).

By looking at terms of order \( \epsilon \) when perturbing \( \gamma \) to \( \gamma + \epsilon v \), we find that

\[
\partial_v \int_\gamma \omega = \int_0^1 \partial_v(\omega_{\gamma(t)}(\gamma'(t)) + \omega_{\gamma(t)}(v'(t))) \, dt
\]

\[
= \int_0^1 \partial_v \omega_1 \cdot \gamma' + \omega \left( \frac{dv}{dt} \right) \, dt
\]

Adding in the boundary terms, we find \( D \log \exp \int \omega = \tau d\omega \), as advertised. \( \square \)

In words, this lemma means we can lower the degree of our forms by moving to pathspace. The primordial example of this is the equivalence between connections (approximately 1-forms on \( X \)) and parallel transport operators (approximately functions on \( \mathcal{P}X \)).
In a perfect world, we could hope that there is some general relationship between 2-forms \( F \) on \( X \) and connection 1-forms \( B \) on \( \cal P X \). The curvature 2-form \( dB + B \wedge B \) on \( \cal P X \) would be related to some 3-form back on \( X \). The vanishing of this 3-form would then be equivalent to the Bianchi identity.

Unfortunately, things aren’t quite so nice. The field copy problem prevents us from constructing a map \( \Omega^2_X(g) \to \Omega^1_{\cal P X}(g) \) in the nonabelian case. The rest of this paper demonstrates a possible resolution to this problem. The trick is to look at degree 2 “forms” with coefficients in a so-called 2-group. These 2-groups remember enough information to let us really construct a map from such objects to \( \Omega^1_{\cal P X} \) in exact analogy with the abelian \( \tau \).

1.3 From Pathspace back to Space

Without going into any details, let us just remark that there is a map \( \epsilon \) which takes “nice enough” \( k \)-forms on \( \cal P X \) and gives us \((k+1)\)-forms on \( X \). \( \epsilon \) works by evaluating the \( k \)-form on an infinitesimally short path to get a \((k+1)\)-form. Antisymmetry follows from parametrization independence.

2 Connections on Pathspace

2.1 Smooth Functions versus Smooth Functors on \( \cal P X \)

If \( G \) is any Lie group and \( \omega \) a \( g \)-valued 1-form on \( X \) and \( \gamma : [0,1] \to X \) a path, we can form the path-ordered product

\[
\exp \int_{\gamma} \omega = \lim_{N \to \infty} \prod_{k=1}^{N} \exp \left\{ \frac{1}{N} \cdot \omega_{\gamma(k/N)} (\gamma'(k/N)) \right\}
\]

This gives the map \( \Omega^1_X(g) \to C^\infty(\cal P X, G) \) which takes a connection to its parallel transport operator. But not every smooth function \( F \) on \( \cal P X \) is a parallel transport \( \longrightarrow F \) must be functorial\(^2\), in the sense that

\[
F(\beta \circ \alpha) = F(\beta) \cdot F(\alpha)
\]

We are therefore only interested in smooth functors from \( \cal P X \) to \( G \), since these are the only sort of thing which may arise from a connection on \( X \). In other words, we are not interested in functions on \( \cal P X \) since they do not respect the extra structure of pathspace. On the other hand, functors respect the underlying composition structure.

\(^2\)If we treat the group \( G \) as a category with one object and all arrows invertible.
What is the infinitessimal version of this functorality? Define
\[ B = F^*\theta = F^{-1} \cdot dF \]
and let \( v \) be a vectorfield on \( \gamma \). Now subdivide \( \gamma = \beta \circ \alpha \), splitting \( v \) into a vectorfield \( x \) on \( \alpha \) and \( y \) on \( \beta \). For shorthand, we will write \( v = y \circ x \). Now if \( F \) is functorial,
\[ d (F_{\beta \circ \alpha}) (y \circ x) = dF_{\beta}(y) \cdot F_\alpha + F_{\beta} \cdot dF_\alpha(x) \]
so we have
\[ B_\gamma(v) = B_{\beta \circ \alpha}(y \circ x) = F_{\beta \circ \alpha}^{-1} \cdot dF_{\beta \circ \alpha}(y \circ x) \]
\[ = F_{\alpha}^{-1} \cdot F_{\beta}^{-1} dF_\beta(y) \cdot F_\alpha + F_{\alpha}^{-1} dF_\alpha(x) \]
\[ = B_\alpha(x) + \text{Ad}(F_{\alpha}^{-1})(B_\beta(y)) \]

Of course, an arbitrary 1-form \( \omega \) on \( \mathcal{P}X \) will not have this property — there will generally be no relationship between \( \omega_\alpha \), \( \omega_\beta \) and \( \omega_{\beta \circ \alpha} \). We want our connections to have this infinitessimal functorality so that parallel transport gives a functor instead of merely a function. This means we need to restrict somewhat our naive notion of connection on pathspace. To ensure functorality, we must have
\[ B_{\beta \circ \alpha}(y \circ x) = B_\alpha(x) + \alpha(A_\alpha)(B_\beta(y)) \]
where \( A \) is some functor with values in \( H \) and \( \alpha \) is an action of \( H \) on \( \mathfrak{g} \).

This leads us to define the connection in two parts:

**Definition 2.1.** Let \( G, H \) be Lie groups equipped with an action
\[ H \xrightarrow{\alpha} \text{Aut}(G) \]
A \((\mathfrak{g}, \mathfrak{h})\)-connection on \( \mathcal{P}X \) is a pair \((B, A)\) with
\[ B \in \Omega^2_{\mathcal{P}X}(\mathfrak{g}) \]
\[ A \in \Omega^1_{\mathcal{P}X}(\mathfrak{h}) \]

\(^3\text{It is interesting to note that if an arbitrary 1-form } \omega \text{ on } \mathcal{P}X \text{ satisfies} \]
\[ d\omega + \omega \wedge \omega = 0 \]
then it automatically is infinitessimally functorial, since the antiderivative \( P \) satisfies
\[ P^{-1}dP_{\beta \circ \alpha} = P^{-1}dP_\alpha + \text{Ad}(P_{\alpha}^{-1})(P^{-1}dP_\beta) \]
This is why the problem of functorality never appears in *A Poincaré Lemma for Connection Forms*. 

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Given a \((\mathfrak{g}, \mathfrak{h})\)-connection, we can construct a \(\mathfrak{g}\)-valued 1-form \(\omega\) on \(PX\) by

\[
\omega_\gamma(v) \overset{\text{def}}{=} \int_0^1 B_{\gamma(t)}(v(t), \gamma'(t)) \exp f_{\gamma} A \, dt
\]

where the superscript represents the action of \(H\) on \(\mathfrak{g}\) and \(\gamma'(t) = \gamma(t \tau)\). The \(\omega\) constructed in this way has the property that

\[
\omega_{\beta \alpha} = \omega_\alpha + (\omega_\beta)^{\alpha}_\alpha
\]

for some element \(g_\alpha \in H\) acting on \(\mathfrak{g}\) and depending functorially on \(\alpha\). In other words, \(\omega\) is infinitesimally functorial. It is not difficult to see that every infinitesimally functorial connection comes from such a construction.

**Example 2.1.** The motivating example comes from choosing \(G = H\) with the adjoint action. Then a \((\mathfrak{g}, \mathfrak{h})\)-connection is given by \(B \in \Omega^2_X(\mathfrak{g}), A \in \Omega^1_X(\mathfrak{g})\). The connection on pathspace is

\[
\omega_\gamma(v) = \int_0^1 P_{\gamma^t} \cdot B(v(t), \gamma'(t)) \cdot P_{\gamma^t}^{-1} \, dr
\]

where

\[
P_{\gamma^t} = \exp \int_{\gamma^t} A
\]

By construction, \(\omega\) satisfies

\[
\omega_{\beta \alpha} = \omega_\alpha + (\omega_\beta)^{\alpha}_\alpha
\]

Direct calculation shows that if \(B + dA + A \wedge A = 0\) then the curvature \(d\omega + \omega \wedge \omega\) vanishes.
2.2 2-Groups

The data $(G, H, \alpha)$ is almost the defining data for a strict 2-group, which is a group with both “horizontal” and “vertical” multiplication (though the vertical operation is only partially-defined, like composition).

**Definition 2.2.** A (strict) 2-group is a quadruple $G = (G, H, t, \alpha)$ where $G$ and $H$ are groups,

$$H \xrightarrow{\alpha} \text{Aut}(G)$$

an action of $H$ on $G$ and

$$G \xrightarrow{t} H$$
a homomorphism. $t$ and $\alpha$ must be compatible in the following two ways: $t$ must intertwine $\alpha$ with the adjoint action on $H$

$$G \xrightarrow{\alpha_h} G$$

$\xrightarrow{t}$

$$H \xrightarrow{\text{Ad}(h)} H$$

and the Peiffer identity

Given a 2-group, we can form the group of arrows $G \rtimes H$. This group has the usual horizontal product

$$(g_2, h_2) \cdot (g_1, h_2) = (g_2 \cdot \alpha(h_2)(g_1), h_2 h_1)$$

but also a vertical product: whenever $h_2 = t(g_1) \cdot h_1$ we can define

$$(g_2, t(g_1)h_1) \circ (g_1, h_1) = (g_2 g_1, h_1)$$

A 2-connection will assign an infinitesimal group element in $g \rtimes h$ to each bit of surface by assigning an element of $h$ to line elements (the “starting point”) and an element of $g$ to surface elements (the “motion along the surface”).
Definition 2.3. A 2-connection $\eta$ on $X$ for the 2-group $G = (G, H, t, \alpha)$ is given by a 2-form $B \in \Omega^2_X(g)$ and a 1-form $A \in \Omega^1_X(h)$. The curvature of $\eta$ is the 3-form
\[ d^A B = dB + A \wedge d\alpha B \in \Omega^3_X(g) \]
where the wedge product is defined using the action $d\alpha$ of $h$ on $g$. The fake curvature of $\eta$ is the 2-form
\[ dA + A \wedge A + t(B) \in \Omega^2_X(h) \]
Every 2-connection on $X$ determines a $g$-valued connection $\omega$ on $P_X$ by
\[ \omega_\gamma(v) = \int_0^1 \alpha(\exp \int_{\gamma(t)} A(v(t), \gamma'(t)) \, dt) \]
Once we have a 2-connection, we can compute surface-ordered products!

Definition 2.4. Let $\eta = (B, A)$ be a 2-connection for $G$ as above, and let $\omega$ be the associated $g$-valued connection on $P_X$. Then to any curve $\Gamma$ in $P_X$ (that is, a parametrized surface in $X$) we can associate the surface-ordered product
\[ \exp \int \int_{\Gamma} \eta \overset{def}{=} \exp \int_{\Gamma} \omega \]
There are other curves $\Gamma'$ in $P_X$ corresponding to the same surface $S$ in $X$. If (and only if) the fake curvature of $\eta$ vanishes, a theorem of Baez and Schreiber shows that
\[ \exp \int \int_{\Gamma} \eta = \exp \int \int_{\Gamma'} \eta \]
so we can define the surface-ordered product
\[ \exp \int \int_{S} \eta \]
without worrying about which parametrization of $S$ is chosen.