Pathspace Connections and 2-Groups

Matt Noonan

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1 Motivation

Throughout these notes, a fixed local trivialization of all principal bundles is assumed. This allows us to simplify some calculations by dealing with $\mathfrak{g}$-valued k-forms and G-valued functions directly.

1.1 Fundamental Theorem of Calculus

Let us first recall why electromagnetism is easy:

Lemma 1.1. *(Poincaré, Gauss, Leibniz, ...) There is an exact sequence of sheaves*

$$1 \longrightarrow \underline{U(1)} \xrightarrow{C} C^\infty(X, U(1)) \xrightarrow{d\log} \Omega^1_X(i\mathbb{R}) \xrightarrow{d} \Omega^2_X(i\mathbb{R}) \xrightarrow{d} \Omega^3_X(i\mathbb{R}) \xrightarrow{d} \dots$$

where $\underline{U(1)}$ is the sheaf of locally constant functions.

In particular, if we are given a 2-form $F \in \Omega^2_X(i\mathbb{R})$, we know that F locally comes from a potential $A \in \Omega^1$ if and only if $dF = 0$. This means we can easily test which 2-forms could be electromagnetic fields, and finding a potential is no more difficult than finding an antiderivative.

For more general Yang-Mills gauge fields, the group $U(1)$ will be replaced with a nonabelian group G. In this case, we at least have

Lemma 1.2. *(Cartan?) There is an exact sequence of sheaves*

$$1 \longrightarrow \underline{G} \xrightarrow{C} C^\infty(X, G) \xrightarrow{{}^*\theta} \Omega^1_X(\mathfrak{g}) \xrightarrow{\text{curv}} \Omega^2_X(\mathfrak{g})$$

where ${}^*\theta : f \mapsto f^{-1} \cdot df$ and $\text{curv}\ \omega = d\omega + \omega \wedge \omega$.

Unfortunately, the obvious necessary condition for $F \in \Omega^2(\mathfrak{g})$ to satisfy $\text{curv}\ \omega = F$ for some $\omega \in \Omega^1(\mathfrak{g})$ is the Bianchi identity

$$0 = d^\omega F = dF + \omega \wedge F$$

which explicitly depends on already having the gauge potential ω on hand.

1.2 From Space to Pathspace

Let $\widetilde{\mathcal{P}}X$ denote the space of $[0,1]$-parametrized, piecewise smooth paths in X. Similarly, let $\mathcal{P}X$ be the space of unparametrized (but oriented) piecewise smooth paths in X.

It is natural to ask what the fundamental theorem of calculus looks like from the perspective of pathspace. If $\omega \in \Omega^1_X(i\mathbb{R})$ is given, then we can get a function $P : \mathcal{P}X \longrightarrow U(1)$ by

$$P(\gamma) = \exp\left(\int_\gamma \omega\right)$$

Likewise, a k-form $\eta \in \Omega^k_X(i\mathbb{R})$ gives a $(k-1)$-form $\tau(\eta) \in \Omega^{k-1}_{\mathcal{P}X}(i\mathbb{R})$ by the formula

$$\tau(\eta)_\gamma(v_1, ..., v_{k-1}) = \int_0^1 \eta(v_1(t), ..., v_{k-1}(t), \gamma'(t))\ dt$$

where the v_i are vectorfields along γ (that is, tangent vectors to $\mathcal{P}X$ at γ). Note that this formula does not depend on the parametrization of γ.

The above discussion can be summarized by this suggestive diagram, where $G = U(1)$, Ω^k means $\Omega^k(\mathfrak{g})$, and C^∞_X means $C^\infty(X, G)$:

$$\begin{tikzcd}
1 \arrow[r] & \underline{G} \arrow[r] \arrow[d] & C^\infty_X \arrow[r, "d\log"] \arrow[d, "i"] & \Omega^1_X \arrow[r, "d"] \arrow[d, "\exp\int"] & \Omega^2_X \arrow[r, "d"] \arrow[d, "\tau"] & \Omega^3_X \arrow[r, "d"] \arrow[d, "\tau"] & \cdots \\
 & 1 \arrow[r] & \underline{G}_{\mathcal{P}} \arrow[r] & C^\infty_{\mathcal{P}X} \arrow[r, "D\log"'] & \Omega^1_{\mathcal{P}X} \arrow[r, "D"'] & \Omega^2_{\mathcal{P}X} \arrow[r, "D"'] & \cdots
\end{tikzcd}$$

Here, $\underline{G}_{\mathcal{P}}$ means “functions on pathspace which only depend on the endpoints”.[1]

Lemma 1.3. *Everything in this diagram commutes.*

Proof. First, we need to explain the map $D : \Omega^k_{\mathcal{P}X} \longrightarrow \Omega^{k+1}_{\mathcal{P}X}$. This is essentially the exterior derivative on pathspace, with a corrective factor thrown in for the motion of the endpoints. Explicitly, if $\omega \in \Omega^k_{\mathcal{P}X}(i\mathbb{R})$ is given and we choose a curve γ, then there is some k-form $\hat{\omega}$ such that

$$\omega_\gamma(v_1, \ldots, v_k) = \int_0^1 \hat{\omega}_{\gamma(t)}(v_1(t), \ldots, v_k(t))\ dt$$

Now we define

$$(D\omega)_\gamma(v_0, ..., v_k) = (d\omega)_\gamma(v_0, ..., v_k) + \widehat{d\omega}_{\gamma(0)}(v_0(0), \ldots, v_k(0)) - \widehat{d\omega}_{\gamma(1)}(v_0(1), \ldots, v_k(1))$$

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[1] Although this definition may seem *ad hoc*, it is motivated by thinking of the pathspace $\mathcal{P}X$ as a sort of smooth category.

Let us just prove that the square

$$\begin{tikzcd}
\Omega^1_X \arrow[r, "d"] \arrow[d, "\exp\int"] & \Omega^2_X \arrow[d, "\tau"] \\
C^\infty_{\mathcal{P}X} \arrow[r, "D\log"'] & \Omega^1_{\mathcal{P}X}
\end{tikzcd}$$

commutes, the general case being just a messier version of this. Starting with an arbitrary $\omega \in \Omega^1_X$, write

$$\omega = \sum_{i=1}^n \omega_i \; dx^i, \qquad d\omega = \frac{1}{2}\sum_{i,j=1}^n \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) \; dx^i \wedge dx^j$$

Now pick a curve γ and a vectorfield $v = v^i \; \partial/\partial x^i$ along it. Without (much) loss of generality, assume that γ' is linearly dependent on $\partial/\partial x^1$. Then we have

$$\begin{aligned}
\tau(d\omega)_\gamma(v) &= \int_0^1 d\omega_{\gamma(t)}(v(t), \gamma'(t)) \; dt \\
&= \sum_{i=1}^n \int_0^1 \left(\frac{\partial \omega_1}{\partial x^i} - \frac{\partial \omega_i}{\partial x_1} \right) \cdot v^i \cdot \gamma' \; dt \\
&= \int_0^1 \partial_v \omega_1 \cdot \gamma' \; dt - \sum_{i=1}^n \int_0^1 \frac{d\omega_i}{dt} \cdot v^i \; dt \\
&= \int_0^1 \partial_v \omega_1 \cdot \gamma' + \omega(\frac{dv}{dt}) \; dt + \omega_{\gamma(0)}(v(0)) - \omega_{\gamma(1)}(v(1))
\end{aligned}$$

On the other hand, we can compute $d\int\omega$ and add in boundary terms to find $D\log\exp\int\omega$. By looking at terms of order ϵ when perturbing γ to $\gamma + \epsilon v$, we find that

$$\begin{aligned}
\partial_v \int_\gamma \omega &= \int_0^1 \partial_{v(t)} \omega_{\gamma(t)}(\gamma'(t)) + \omega_{\gamma(t)}(v'(t)) \; dt \\
&= \int_0^1 \partial_v \omega_1 \cdot \gamma' + \omega(\frac{dv}{dt}) \; dt
\end{aligned}$$

Adding in the boundary terms, we find $D\log\exp\int\omega = \tau d\omega$, as advertised. □

In words, this lemma means we can lower the degree of our forms by moving to pathspace. The primordial example of this is the equivalence between connections (approximately 1-forms on X) and parallel transport operators (approximately functions on $\mathcal{P}X$).

In a perfect world, we could hope that there is some general relationship between 2-forms F on X and connection 1-forms B on $\mathcal{P}X$. The curvature 2-form $dB + B \wedge B$ on $\mathcal{P}X$ would be related to some 3-form back on X. The vanishing of this 3-form would then be equivalent to the Bianchi identity.

Unfortunately, things aren't quite so nice. The field copy problem prevents us from constructing a map $\Omega^2_X(\mathfrak{g}) \xrightarrow{\tau} \Omega^1_{\mathcal{P}X}(\mathfrak{g})$ in the nonabelian case. The rest of this paper demonstrates a possible resolution to this problem. The trick is to look at degree 2 "forms" with coefficients in a so-called 2-group. These 2-groups remember enough information to let us really construct a map from such objects to $\Omega^1_{\mathcal{P}X}$ in exact analogy with the abelian τ.

1.3 From Pathspace back to Space

Without going into any details, let us just remark that there is a map ϵ which takes "nice enough" k-forms on $\mathcal{P}X$ and gives us $(k+1)$-forms on X. ϵ works by evaluating the k-form on an infinitessimally short path to get a $(k+1)$-form. Antisymmetry follows from parametrization independence.

2 Connections on Pathspace

2.1 Smooth Functions versus Smooth Functors on $\mathcal{P}X$

If G is any Lie group and ω a $\mathfrak{g}$-valued 1-form on X and $\gamma : [0,1] \longrightarrow X$ a path, we can form the path-ordered product

$$\exp \int_\gamma \omega \overset{\text{def}}{=} \lim_{N\to\infty} \prod_{k=1}^{N} \exp\left\{ \frac{1}{N} \cdot \omega_{\gamma(k/N)}\left(\gamma'(k/N)\right) \right\}$$

This gives the map $\Omega^1_X(\mathfrak{g}) \longrightarrow C^\infty(\mathcal{P}X, G)$ which takes a connection to its parallel transport operator. But not every smooth function F on $\mathcal{P}X$ is a parallel transport — F must be functorial[2], in the sense that

$$F(\beta \circ \alpha) = F(\beta) \cdot F(\alpha)$$

We are therefore only interested in smooth *functors* from $\mathcal{P}X$ to G, since these are the only sort of thing which may arise from a connection on X. In other words, we are not interested in *functions* on $\mathcal{P}X$ since they do not respect the extra structure of pathspace. On the other hand, *functors* respect the underlying composition structure.

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[2] If we treat the group G as a category with one object and all arrows invertible.

What is the infinitessimal version of this functorality? Define

$$B = F^*\theta = F^{-1} \cdot dF$$

and let v be a vectorfield on γ. Now subdivide $\gamma = \beta \circ \alpha$, splitting v into a vectorfield x on α and y on β. For shorthand, we will write $v = y \circ x$. Now if F is functorial,

$$d\left(F_{\beta\circ\alpha}\right)(y \circ x) = dF_\beta(y) \cdot F_\alpha + F_\beta \cdot dF_\alpha(x)$$

so we have

$$\begin{aligned} B_\gamma(v) = B_{\beta\circ\alpha}(y \circ x) &= F_{\beta\circ\alpha}^{-1} \cdot dF_{\beta\circ\alpha}(y \circ x) \\ &= F_\alpha^{-1} \cdot F_\beta^{-1} dF_\beta(y) \cdot F_\alpha + F_\alpha^{-1} dF_\alpha(x) \\ &= B_\alpha(x) + \mathrm{Ad}(F_\alpha^{-1})(B_\beta(y)) \end{aligned}$$

Of course, an arbitrary 1-form ω on $\mathcal{P}X$ will not have this property — there will generally be no relationship between ω_α, ω_β and $\omega_{\beta\circ\alpha}$[3]. We want our connections to have this infinitessimal functorality so that parallel transport gives a *functor* instead of merely a *function*. This means we need to restrict somewhat our naïve notion of connection on pathspace. To ensure functorality, we must have

$$B_{\beta\circ\alpha}(y \circ x) = B_\alpha(x) + \alpha(A_\alpha)(B_\beta(y))$$

where A is some functor with values in H and α is an action of H on $\mathfrak{g}$.

This leads us to define the connection in two parts:

Definition 2.1. Let G, H be Lie groups equipped with an action

$$H \xrightarrow{\alpha} Aut(G)$$

A $(\mathfrak{g}, \mathfrak{h})$-*connection* on $\mathcal{P}X$ is a pair (B, A) with

$$\begin{aligned} B &\in \Omega^2_{\mathcal{P}X}(\mathfrak{g}) \\ A &\in \Omega^1_{\mathcal{P}X}(\mathfrak{h}) \end{aligned}$$

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[3] It is interesting to note that if an arbitrary 1-form ω on $\mathcal{P}X$ satisfies

$$d\omega + \omega \wedge \omega = 0$$

then it automatically is infinitessimally functorial, since the antiderivative P satisfies

$$P^{-1}dP_{\beta\circ\alpha} = P^{-1}dP_\alpha + \mathrm{Ad}(P_\alpha^{-1})(P^{-1}dP_\beta)$$

This is why the problem of functorality never appears in *A Poincaré Lemma for Connection Forms.*

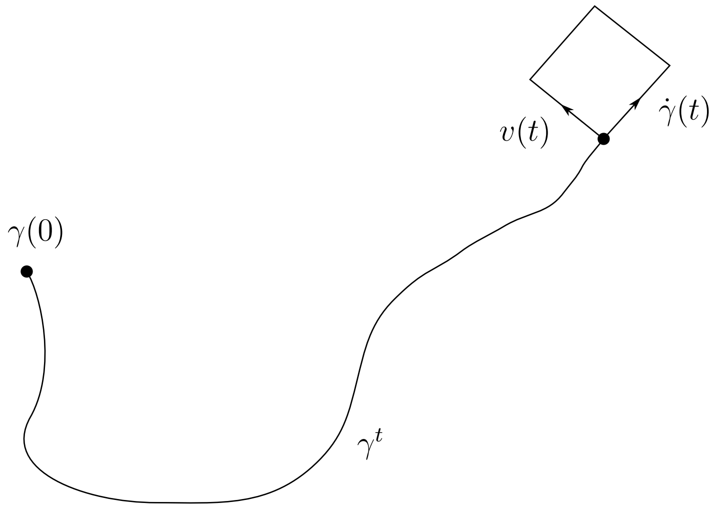

Figure 1: Schematic depiction of $B_{\gamma(t)}(v(t),\dot\gamma(t))^{\exp\int_{\gamma^t}A}$

Given a $(\mathfrak{g},\mathfrak{h})$-connection, we can construct a $\mathfrak{g}$-valued 1-form ω on $\mathcal{P}X$ by

$$\omega_\gamma(v) \overset{\text{def}}{=} \int_0^1 B_{\gamma(t)}(v(t),\gamma'(t))^{\exp\int_{\gamma^t}A}\,dt$$

where the superscript represents the action of H on $\mathfrak{g}$ and $\gamma^t(\tau)=\gamma(t\tau)$. The ω constructed in this way has the property that

$$\omega_{\beta\circ\alpha} = \omega_\alpha + (\omega_\beta)^{g_\alpha}$$

for some element $g_\alpha \in H$ acting on $\mathfrak{g}$ and depending functorially on α. In other words, ω is infinitessimally functorial. It is not difficult to see that every infinitessimally functorial connection comes from such a construction.

Example 2.1. The motivating example comes from choosing $G=H$ with the adjoint action. Then a $(\mathfrak{g},\mathfrak{h})$-connection is given by $B\in\Omega^2_X(\mathfrak{g})$, $A\in\Omega^1_X(\mathfrak{g})$. The connection on pathspace is

$$\omega_\gamma(v) = \int_0^1 P_{\gamma^t}\cdot B(v(t),\gamma'(t))\cdot P_{\gamma^t}^{-1}\,dr$$

where

$$P_{\gamma^t} = \exp\int_{\gamma^t} A$$

By construction, ω satisfies

$$\omega_{\beta\circ\alpha} = \omega_\alpha + (\omega_\beta)^{P_\alpha}$$

Direct calculation shows that if $B + dA + A\wedge A = 0$ then the curvature $d\omega + \omega\wedge\omega$ vanishes.

2.2 2-Groups

The data (G, H, α) is almost the defining data for a strict 2-group, which is a group with both "horizontal" and "vertical" multiplication (though the vertical operation is only partially-defined, like composition).

Definition 2.2. A *(strict) 2-group* is a quadruple $\mathsf{G} = (G, H, t, \alpha)$ where G and H are groups,

$$H \xrightarrow{\alpha} Aut(G)$$

an action of H on G and

$$G \xrightarrow{t} H$$

a homomorphism. t and α must be compatible in the following two ways: t must intertwine α with the adjoint action on H

$$\begin{tikzcd}
G \arrow[r, "\alpha_h"] \arrow[d, "t"'] & G \arrow[d, "t"] \\
H \arrow[r, "\mathrm{Ad}(h)"'] & H
\end{tikzcd}$$

and the Peiffer identity

$$\begin{tikzcd}
G \arrow[d, "t"'] \arrow[rd, "Ad"] & \\
H \arrow[r, "\alpha"'] & Aut(G)
\end{tikzcd}$$

must hold.

Given a 2-group, we can form the *group of arrows* $G \rtimes_\alpha H$. This group has the usual horizontal product

$$(g_2, h_2) \cdot (g_1, h_2) = (g_2 \cdot \alpha(h_2)(g_1), h_2 h_1)$$

but also a vertical product: whenever $h_2 = t(g_1) \cdot h_1$ we can define

$$(g_2, t(g_1)h_1) \circ (g_1, h_1) = (g_2 g_1, h_1)$$

A 2-connection will assign an infinitessimal group element in $\mathfrak{g} \rtimes \mathfrak{h}$ to each bit of surface by assigning an element of $\mathfrak{h}$ to line elements (the "starting point") and an element of $\mathfrak{g}$ to surface elements (the "motion along the surface").

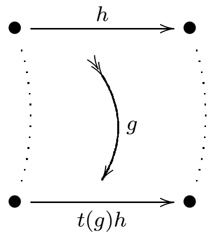

Figure 2: $(g, h) \in \mathfrak{g} \rtimes \mathfrak{h}$ Acting on a Surface Element

Definition 2.3. A *2-connection* η on X for the 2-group $\mathsf{G} = (G, H, t, \alpha)$ is given by a 2-form $B \in \Omega^2_X(\mathfrak{g})$ and a 1-form $A \in \Omega^1_X(\mathfrak{h})$. The *curvature* of η is the 3-form

$$d^A B = dB + A \wedge_{d\alpha} B \in \Omega^3_X(\mathfrak{g})$$

where the wedge product is defined using the action $d\alpha$ of $\mathfrak{h}$ on $\mathfrak{g}$. The *fake curvature* of η is the 2-form

$$dA + A \wedge A + t(B) \in \Omega^2_X(\mathfrak{h})$$

Every 2-connection on X determines a $\mathfrak{g}$-valued connection ω on $\mathcal{P}X$ by

$$\omega_\gamma(v) = \int_0^1 \alpha(\exp \int_{\gamma^t} A)\Big(B(v(t), \gamma'(t)\Big)\ dt$$

Once we have a 2-connection, we can compute surface-ordered products!

Definition 2.4. Let $\eta = (B, A)$ be a 2-connection for G as above, and let ω be the associated $\mathfrak{g}$-valued connection on $\mathcal{P}X$. Then to any curve Γ in $\mathcal{P}X$ (that is, a parametrized surface in X) we can associate the *surface-ordered product*

$$\exp \iint_\Gamma \eta \overset{def}{=} \exp \int_\Gamma \omega$$

There are other curves Γ' in $\mathcal{P}X$ corresponding to the same surface S in X. If (and only if) the fake curvature of η vanishes, a theorem of Baez and Schreiber shows that

$$\exp \iint_\Gamma \eta = \exp \iint_{\Gamma'} \eta$$

so we can define the surface-ordered product

$$\exp \iint_S \eta$$

without worrying about which parametrization of S is chosen.