Notes on 2-Groups

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1 Basic Definitions

1.1 Top-Down

The cocktail party definition of a 2-group is a "category where everything is a group". To unroll this definition, we first need a diagrammatic description of the axioms for a category.

A *small category* C consists of a set O of "objects" and a set A of "arrows", along with the following functions:

1. To each object we can associate an "identity arrow", so we have a map

$$O \xrightarrow{\mathbf{1}} A$$

2. There are two maps, called "source" and "target", from A to O:

$$\begin{tikzcd} A \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & O \end{tikzcd}$$

3. There is a map called "composition" :

$$A_s \times_t A \xrightarrow{\circ} A$$

   where $A_s \times_t A$ is the fibered product over s and t.

These functions must in turn satisfy certain relationships:

1. The identity arrow is a right inverse to both s and t:

$$s \circ \mathbf{1} = t \circ \mathbf{1} = \mathrm{id}$$

2. For all arrows $\alpha \in A$, we have
$$\mathbf{1}(t(\alpha)) \circ \alpha = \alpha \circ \mathbf{1}(s(\alpha)) = \alpha$$
3. For all arrows $f, g, h \in A$ such that the composition $(h \circ g) \circ f$ is well-defined,
$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dr, "g\circ f"'] & \bullet \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\
 & \bullet \arrow[r, "h"] & \bullet
\end{tikzcd}$$
is a commutative diagram.

Now if C is any category with fibered products, we can define a *category internalized in* C by replacing the word "set" in the definition of small category with the phrase "object of C", and the word "function" with "arrow in C". This lets us write a top-down definition of a 2-group.

Definition 1.1. A *(strict) 2-group* is a category internalized in Groups. In other words, it is a category G such that

1. The objects of G form a group.
2. The arrows of G form a group.
3. The source, target, identity and composition maps are all homomorphisms.

1.2 Examples of 2-groups

Example 1.1. Let G be a group. We define the automorphism 2-group Aut_G as follows: The objects of Aut_G are automorphisms of G, and there is an arrow
$$\phi \xrightarrow{g} \psi$$
whenever $\psi = \mathrm{Ad}(g) \circ \phi$. Each arrow is determined uniquely by a pair $(\phi, g) \in Aut(G) \times G$ corresponding to
$$\phi \xrightarrow{g} \mathrm{Ad}(g) \circ \phi$$
(though the group of arrows is generally not the direct product $Aut(G) \times G$), and we have
$$s(\phi, g) = \phi$$
$$t(\phi, g) = \mathrm{Ad}(g) \circ \phi$$
$$\mathbf{1}(\phi) = (\phi, 1)$$
$$(\mathrm{Ad}(g) \circ \phi, h) \circ (\phi, g) = (\phi, hg)$$

Example 1.2. Every group G is a 2-group such that the group of objects is just $\{1\}$.

Example 1.3. Let G be a group. There is a unique 2-group K_G such that the object group is G and $\mathrm{Hom}(x, y)$ is a single arrow for all $x, y \in G$. The arrow group of K_G is then the semidirect product $G \ltimes_{\mathrm{Ad}} G$. An element $(g, h) \in G \ltimes_{\mathrm{Ad}} G$ determines the arrow

$$h \xrightarrow{g} gh$$

so we have

$$\begin{aligned}
(g_2, h_2) \cdot (g_1, h_1) &= (g_2 h_2 g_1 h_2^{-1}, h_2 h_1) \\
s(g, h) &= h \\
t(g, h) &= gh \\
\mathbf{1}(h) &= (1, h) \\
(g_2, g_1 h_1) \circ (g_1, h_1) &= (g_2 g_1, h_1)
\end{aligned}$$

Example 1.4. Let $G \xrightarrow{\pi} H$ be a covering of Lie groups. We now construct the covering 2-group $\mathsf{Cover}_{G \xrightarrow{\pi} H}$.

The objects of $\mathsf{Cover}_{G \xrightarrow{\pi} H}$ are the points of H. There is one arrow from x to y for each homotopy class of paths in G which project to paths from x to y in H. Composition is concatenation of paths, and multiplication is done by choosing representative curves, multiplying pointwise, and taking the resulting homotopy class.

Example 1.5. Suppose that G is a 2-group with object group G_0 and arrow group G_1, and $H \lhd \mathrm{Hom}(1, 1) \leq G_1$. Then there is a 2-group G/H with the same objects as G and arrow group G_1/H.

Example 1.6. If G is a 2-group as above, and $H \lhd G_0$ then there is a 2-group $\hat{\mathsf{G}}$ with $\hat{G}_0 = G_0$ and

$$\mathrm{Hom}_{\hat{\mathsf{G}}}(x, y) = \begin{cases} \mathrm{Hom}_{\mathsf{G}}(x, y) & \text{if } x = y \bmod H \\ \varnothing & \text{if } x \neq y \bmod H \end{cases}$$

$\hat{\mathsf{G}}$ has one connected component for each element of H.

2 The Exchange Law

Lemma 2.1. *(Eckmann-Hilton argument): Let G be a set equipped with two products $\cdot, \circ$ such that G forms a group under each. If $\cdot, \circ$ share the same identity and satisfy the exchange law*

$$(a \circ b) \cdot (c \circ d) = (a \cdot c) \circ (b \cdot d)$$

then the two products coincide and are abelian.

Proof. To show that the products coincide,

$$x \circ y = (x \cdot 1) \circ (1 \cdot y) = (x \circ 1) \cdot (1 \circ y) = x \cdot y$$

Once we know that the two products coincide, we also have

$$x \circ y = (1 \cdot x) \circ (y \cdot 1) = (1 \circ y) \cdot (x \circ 1) = y \cdot x = y \circ x$$

□

Now suppose that x, y, z, w are arrows in a 2-group G such that $x \circ y$ and $z \circ w$ are defined. Let us temporarily write composition in function notation, so $f \circ g \equiv \circ(f, g)$. The fibered product in Groups is given by

$$A_f \times_g B = \{(a, b) \in A \times B : f(a) = g(b)\} \leq A \times B$$

so that fact that $\circ$ is a homomorphism reads

$$\begin{aligned} (x \circ y) \cdot (z \circ w) &\equiv \circ(x, y) \cdot \circ(z, w) \\ &= \circ\left((x, y) \cdot (z, w)\right) \\ &= \circ(x \cdot z, y \cdot w) \equiv (x \cdot z) \circ (y \cdot w) \end{aligned}$$

In the case of 2-groups the Eckmann-Hilton argument does not apply since $\circ$ and $\cdot$ do not share an identity element. However, the subgroup $\mathrm{Hom}(1, 1) \lhd G_1$ *does* have the same identity for both composition and multiplication. It must therefore be abelian, and composition in $\mathrm{Hom}(1, 1)$ coincides with multiplication.

What's more, the exchange law shows that if G is connected then $\mathrm{Hom}(1, 1)$ is actually a subgroup of the center $Z(G_1)$. First, note that if $x \in \mathrm{Hom}(1, 1)$ and $1 \xrightarrow{f} g$ then

$$f \circ x = (f \cdot 1) \circ (1 \cdot x) = (f \circ 1) \cdot (1 \circ x) = f \cdot x$$

Now let $f \in \mathrm{Hom}(1, -)$ be given. Then

$$f \cdot x \cdot f^{-1} = (f \circ x) \cdot (f^{-1} \circ 1) = (f \cdot f^{-1}) \circ (x \cdot 1) = x$$

so $x \in Z(G_1)$!

2.1 Bottom-Up

There is a second way to define 2-groups which dates from the mid–20th century study of higher homotopy groups. In this context, 2-groups go by the name of "crossed modules". The equivalence of the two definitions of 2-group will be proven in the next section.

Definition 2.1. A *crossed module* is a pair of groups K, H with an action

$$H \xrightarrow{\alpha} Aut(K)$$

and a homomorphism

$$K \xrightarrow{t_0} H$$

such that t_0 intertwines α and the adjoint action on H:

$$\begin{tikzcd} K \arrow[r, "\alpha(h)"] \arrow[d, "t_0"'] & K \arrow[d, "t_0"] \\ H \arrow[r, "\mathrm{Ad}(h)"] & H \end{tikzcd}$$

and satisfies the *Peiffer identity*

$$\begin{tikzcd} K \arrow[d, "t_0"'] \arrow[dr, "\mathrm{Ad}"] & \\ H \arrow[r, "\alpha"'] & Aut(K) \end{tikzcd}$$

Example 2.1. Let G be any group, and set $K = G$, $H = Aut(G)$ with $\alpha(\phi) = \phi$ and $t_0 = \mathrm{Ad}$. t_0 is an intertwiner since for any $\phi \in H$

$$t_0(\alpha(\phi)(g))(g') = t_0(\phi(g))(g') = \phi(g) \cdot g' \cdot \phi(g^{-1})$$

and

$$\mathrm{Ad}(\phi)(t_0(g))(g') = \phi(g \cdot \phi(g'^{-1}) \cdot g^{-1}) = \phi(g) \cdot g' \cdot \phi(g^{-1})$$

The Peiffer identity holds since

$$\alpha(t_0(g))(g') = t_0(g)(g') = \mathrm{Ad}(g)(g')$$

Example 2.2. Let $K = H = G$ for some group G, and set $\alpha = \mathrm{Ad}$, $t_0 = \mathrm{id}$. This corresponds to the 2-group with exactly one arrow between any two objects described in the last section. The intertwining property is immediate, and the Peiffer identity reads:

$$\alpha(t_0(g))(g') = \mathrm{Ad}(t_0(g))(g') = \mathrm{Ad}(g)(g')$$

Example 2.3. The notion of a crossed module came out of MacLane and Whitehead's study of the the relative homotopy group $\pi_2(X, Y)$. Recall that if $Y \subset X$ is a subcomplex then there is a long exact homotopy sequence

$$\ldots \longrightarrow \pi_2(X, Y) \xrightarrow{\partial} \pi_1(Y) \longrightarrow \pi_1(X) \longrightarrow \pi_1(X, Y) \longrightarrow \ldots$$

If $S \in \pi_2(X,Y)$, $\partial(S)$ is just the loop in Y obtained by forgetting the 2-cell in S. $\pi_1(Y)$ acts on $\pi_2(X,Y)$ by adding a "tail".

We can now make a crossed module $(\pi_2(X,Y), \pi_1(Y), \partial,$ "add a tail$'')$. The intertwining property is again immediate, and the Peiffer identity is

$$\alpha(\partial S_2)(S_1) = \mathrm{Ad}(S_2)(S_1)$$

which holds because the 2-cell part of S_2 cancels with the 2-cell part of S_2^{-1}, leaving only the action of the curve $S_2 \cap Y$ on S_1.

We conclude this section with two lemmas about the structure of crossed modules.

Lemma 2.2. *Given a crossed module (K, H, α, t_0), there is a unique way to extend t_0 to the semidirect product*

$$G = K \rtimes_\alpha H$$

such that the extension t acts as the identity on H and as t_0 on K.

Proof. Define the extension $G \xrightarrow{t} H$ by

$$t(g,h) = t_0(g) \cdot h$$

Then t is a homomorphism:

$$\begin{aligned} t\left((g_2,h_2)\cdot(g_1,h_1)\right) &= t\left((g_2 \cdot \alpha(h_2)(g_1), h_2h_1\right) \\ &= t_0(g_2 \cdot \alpha(h_2)(g_1)) \cdot h_2h_1 \\ &= t_0(g_2) \cdot t_0(\alpha(h_2)(g_1)) \cdot h_2h_1 \\ &= t_0(g_2)h_2t_0(g_1)h_2^{-1} \cdot h_2h_1 \\ &= t(g_2,h_2) \cdot t(g_1,h_1) \end{aligned}$$

where the second–to–last equation follows from the intertwining property of t_0. ☐

Lemma 2.3. *If (K, H, α, t_0) is a crossed module, then $K \cong N \oplus A$ for some abelian group A and some normal subgroup $N \lhd H$.*

Proof. First, let $N = t_0(K)$ be the image of t_0. Then $N \lhd H$ since, by the intertwining property,

$$ht_0(g)h^{-1} = t_0(\alpha(h)(g))$$

Now the lemma amounts to showing that there is a central subgroup $A \leq K$ with $K/A \cong N$. Let $A \lhd K$ be the kernel of t_0. Then by the Peiffer identity, for all $g \in K$

$$[a,g] = \mathrm{Ad}(a)(g) \cdot g^{-1} = \alpha(t_0(a))(g) \cdot g^{-1} = g \cdot g^{-1} = 1$$

☐

2.2 Equivalence

We begin with the top-down definition of a 2-group and derive the bottom-up definition. Throughout G will be our 2-group, with object group G_0 and arrow group G_1.

Lemma 2.4. *Let H be the set of objects in G_0 connected to 1. Then $H \lhd G_0$.*

Proof. To see that H is a subgroup, fix two elements $h_1, h_2 \in H$. Since these elements are connected to 1, we can find arrows ψ_1, ψ_2 with sources at 1 and targets h_1, h_2 repsectively. Then we have

$$s(\psi_1 \cdot \psi_2) = 1$$

$$t(\psi_1 \cdot \psi_2) = h_1 \cdot h_2$$

and

$$t(\psi_1^{-1}) = h_1^{-1}$$

so H is closed under multiplication and inversion.

To see that H is normal, fix an element $h \in H$. As before, choose a $\psi \in G_1$ such that $s(\psi) = 1$ and $t(\psi) = h$. If $o \in G_0$ is arbitrary, we compute

$$s(\mathbf{1}_o \cdot \psi \cdot \mathbf{1}_o^{-1}) = o \cdot s(\psi) \cdot o^{-1} = 1$$

and

$$t(\mathbf{1}_o \cdot \psi \cdot \mathbf{1}_o^{-1}) = oho^{-1}$$

Since $\mathbf{1}_o \cdot \psi \cdot \mathbf{1}_o^{-1}$ connects oho^{-1} to the identity, H is normal in G_0. □

To simplify the presentation, for the rest of this section we will assume that the 2-group is connected in the sense that $H = G_0$.

To construct the crossed module associated to a 2-group, begin by considering the subgroup $K = \mathrm{Hom}(1, -) \leq G_1$ of arrows out of 1. This gives the split exact sequence

$$\begin{tikzcd}1 \arrow[r] & K \arrow[r] & G_1 \arrow[r, "s"] & H \arrow[l, bend left, "\mathbf{1}"] \arrow[r] & 1\end{tikzcd}$$

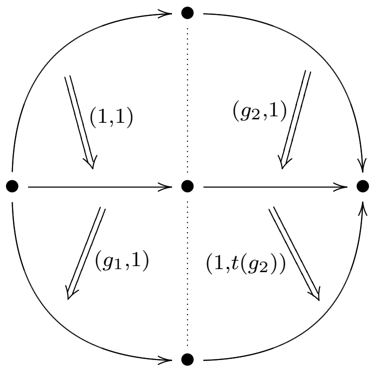
presenting G_1 as a semidirect product

$$G_1 = K \rtimes_\alpha H$$

for the action $\alpha = \mathrm{Ad} \circ \mathbf{1} : H \longrightarrow Aut(K)$. The target map t_0 is obviously just t restricted to K.

Theorem 2.5. *The quadruple (K, H, t, α) is a crossed module.*

Proof. We begin by proving the Peiffer identity. Let $g_1, g_2 \in K$ be arbitrary, and consider the arrows



We can compute this 2-group element in two different ways, which are identified by the exchange law:

$$\big((1, t(g_2)) \circ (g_2, 1)\big) \cdot \big((g_1, 1) \circ (1, 1)\big) = (g_2, 1) \cdot (g_1, 1) = (g_2 \cdot g_1, 1)$$

$$\big((1, t(g_2)) \cdot (g_1, 1)\big) \circ \big((g_2, 1) \cdot (1, 1)\big) = (\alpha(t(g_2))(g_1) \cdot g_2, 1)$$

which implies the Peiffer identity

$$\alpha(t(g_2))(g_1) = \mathrm{Ad}(g_2)(g_1)$$

The intertwining property of $\alpha = \mathrm{Ad} \circ \mathbf{1}$ is more elementary. Let $h \in H$ and $g \in K$ be given. Then

$$t(\alpha(h)(g)) = t(\mathbf{1}_h \cdot g \cdot \mathbf{1}_h^{-1}) = \mathrm{Ad}\big(t(\mathbf{1}_h)\big)t(g) = \mathrm{Ad}(h)t(g)$$

since the target of $\mathbf{1}_h$ is just h. This completes the proof. □

Now we reverse the argument to construct a 2-group from a crossed module. Again, assume that the crossed module (K, H, t_0, α) is connected (that is, t_0 is surjective). Define $G_1 = K \rtimes_\alpha H$, $G_0 = H$, and for any $(g, h) \in G_1$

$$\begin{aligned} s(g,h) &= h \\ t(g,h) &= t_0(g) \cdot h \\ \mathbf{1}_h &= (1, h) \\ (g_2, t(g_1) \cdot h) \circ (g_1, h) &= (g_2 \cdot g_1, h) \end{aligned}$$

Theorem 2.6. *The data given above defines a 2-group.*

Proof. We proved that t is a homomorphism in the last section, so all that remains is to prove that $\circ$ is a homomorphism and check the axioms of a category.

To see that $\circ$ is a homomorphism, it is enough to check that the exchange law holds. Pick $g_1, \ldots, g_4, h_1$, and h_2 arbitrary and compute

$$\begin{aligned}\big((g_4, t(g_2)h_2) \circ (g_2, h_2)\big) \cdot \big((g_3, t(g_1)h_1) \circ (g_1, h_1)\big) &= (g_4 g_2, h_2) \cdot (g_3 g_1, h_1) \\ &= (g_4 g_2 \cdot \alpha(h_2)(g_3 g_1), h_2 h_1)\end{aligned}$$

$$\begin{aligned} & \big((g_4, t(g_2)h_2) \cdot (g_3, t(g_1)h_1)\big) \circ \big((g_2, h_2) \cdot (g_1, h_1)\big) = \\ = \;& (g_4 \alpha(t(g_2)h_2)(g_3), t(g_2)h_2 t(g_1)h_1) \circ (g_2 \alpha(h_2)(g_1), h_2 h_1) \\ = \;& (g_4 g_2 \alpha(h_2)(g_3) g_2^{-1} \cdot g_2 \alpha(h_2)(g_1), h_2 h_1) \\ = \;& (g_4 g_2 \cdot \alpha(h_2)(g_3 g_1), h_2 h_1)\end{aligned}$$

where the second-to-last equality is an application of the Peiffer identity. This verifies the exchange law, so $\circ$ is a homomorphism $G_{1s} \times_t G_1 \xrightarrow{\circ} G_1$.

Now we verify the category axioms. $\mathbf{1}$ is a right identity for s and t since

$$\begin{aligned} s(\mathbf{1}(h)) &= s(1, h) = h \\ t(\mathbf{1}(h)) &= t(1, h) = t_0(1) \cdot h = h \end{aligned}$$

Next, let us verify that $\mathbf{1}$ acts as an identity for composition. If $(g, h) \in G_1$ is arbitrary,

$$(g, h) \circ \mathbf{1}(s(g, h)) = (g, h) \circ (1, h) = (g, h)$$

$$\mathbf{1}(t(g, h)) \circ (g, h) = (1, t(g) \cdot h) \circ (g, h) = (g, h)$$

Finally, verify the associativity of $\circ$:

$$\Bigg((g_3, t(g_2 g_1) \cdot h) \circ \Big((g_2, t(g_1)h) \circ (g_1, h)\Big)\Bigg) = \big((g_3, t(g_2 g_1) \cdot h) \circ (g_2 g_1, h)\big) = (g_3 g_2 g_1, h)$$

$$\Bigg(\Big((g_3, t(g_2 g_1) \cdot h) \circ (g_2, t(g_1)h)\Big) \circ (g_1, h)\Bigg) = \big((g_3 g_2, t(g_1) \cdot h) \circ (g_1, h)\big) = (g_3 g_2 g_1, h)$$

□

This completes the proof that "categories internal to Groups" and "crossed modules" are identical, at least in the connected case. The non-connected case is almost identical, but there is an extra action of the group of components on $\mathrm{Hom}(1, 1)$ to account for.