Geometric Realizations of $H^3(X, \mathbb{Z})$

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1 Projective Hilbert Space Bundles

Throughout this section, $\mathcal{H}$ will denote an infinite-dimensional separable Hilbert space.

1.1 The Topological Classification

The conceptually simplest way of constructing a geometric realization of $H^3(X, \mathbb{Z})$ is by classifying projective Hilbert space bundles over $X$ (that is, bundles with fiber $\mathbb{P}H$).

**Lemma 1.1.** The unit sphere $S \subset \mathcal{H}$ is contractible.

**Proof.** The lemma is easiest to show in $\mathcal{H} = L^2([0, 1], dx)$. Let $f \in S \subset \mathcal{H}$ be given, and define

$$h_t(f)(x) = \begin{cases} 1 & \text{if } x < t \\ f(t + (1-t)x) & \text{if } x \geq t \end{cases}$$

$h_t$ clearly takes $S$ to itself, $h_0 = \text{id}$ and $h_1 : f \mapsto 1$, which completes the proof. \(\square\)

Let $GL(\mathcal{H})$ be the group of bounded linear operators on $\mathcal{H}$ with bounded inverses.

**Theorem 1.2.** $GL(\mathcal{H})$ is contractible.

**Proof.** First fix a filtration of

$$\mathcal{H} = \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \ldots$$

so $\mathcal{H}_k/\mathcal{H}_{k+1}$ is 1-dimensional. Using this filtration, we can write any $T \in GL(\mathcal{H})$ as

$$T = U \cdot \Delta$$

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where $\Delta$ is “upper triangular” with respect to the filtration and $U$ is unitary. Since the upper triangular operators are contractible, the problem is reduced to understanding the topology of $U(\mathcal{H})$.

Now consider the coset space $U(\mathcal{H}_k)/U(\mathcal{H}_{k+1})$. This is just the tautological $U(1)$-bundle over the Grassmannian of codimension-1 subspaces of $\mathcal{H}_k$. Since this is isomorphic to the unit sphere $S$, $U(\mathcal{H}_k)$ retracts onto $U(\mathcal{H}_{k+1})$. Continuing up the filtration retracts $U(\mathcal{H})$ to the identity.

We want to classify $\mathbb{P}_\mathcal{H}$-bundles over some finite-dimensional manifold $X$. The transition functions for a $\mathbb{P}_\mathcal{H}$-bundle lie in the group $PGL(\mathcal{H})$, isomorphic to $GL(\mathcal{H})/\mathbb{C}^*$. This leads to the exact sheaf sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow GL(\mathcal{H}) \longrightarrow PGL(\mathcal{H}) \longrightarrow 1$$

of the relevant structure groups, yielding the long exact sequence

$$\ldots \longrightarrow H^1(X, GL(\mathcal{H})) \longrightarrow H^1(X, PGL(\mathcal{H})) \overset{\partial}{\longrightarrow} H^2(X, \mathbb{C}^*) \longrightarrow H^2(X, GL(\mathcal{H})) \longrightarrow \ldots$$

in cohomology. Since $GL(\mathcal{H})$ is contractible any $GL(\mathcal{H})$-bundle has a section, so $H^1(X, GL(\mathcal{H})) = 1$. Likewise$^1$, $H^2(X, GL(\mathcal{H})) = 1$ so we have an isomorphism

$$1 \longrightarrow H^1(X, GL(\mathcal{H})) \overset{\sim}{\longrightarrow} H^2(X, \mathbb{C}^*) \longrightarrow 1$$

Combining this with the exponential sequence $\mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^*$ gives an isomorphism

$$H^1(X, GL(\mathcal{H})) \cong H^2(X, \mathbb{C}^*) \cong H^3(X, \mathbb{Z})$$

In other words, projective Hilbert space bundles are topologically classified by $H^3(X, \mathbb{Z})$.

Unfortunately, it is not very easy to construct a $\mathbb{P}_\mathcal{H}$-bundle with a given class $\alpha \in H^3(X, \mathbb{Z})$.

### 1.2 Why $PGL(\mathcal{H})$ Bundles?

Let $G$ be a group, and recall that a $K(G, n)$ is a topological space $X$ such that

$$\pi_k(X) \cong \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

$^1$At least, given the right definition for $H^2(X, GL(\mathcal{H}))$. We know that multiplication in $GL(\mathcal{H})$ is abelian up to homotopy, so if we require the Čech cochains to match only up to homotopy, $H^2(X, GL(\mathcal{H}))$ makes sense and is indeed trivial. This follows in general from the fact that $GL(\mathcal{H})$ is the total space of the universal $\mathbb{C}^*$-bundle over $K(\mathbb{Z}, 2)$.  

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Of course when \( n \geq 2 \), \( G \) must be abelian.

Let us consider the spaces \( K(\mathbb{Z}, n) \) for small \( n \). When \( n = 0 \), \( \mathbb{Z} \) itself is \( K(\mathbb{Z}, 0) \). When \( n = 1 \), we can take the group \( U(1) \) to be \( K(\mathbb{Z}, 1) \). Any \( U(1) \)-bundle \( L \in H^1(X, U(1)) \) is topologically determined by its Chern class \( \delta L \in H^2(X, \mathbb{Z}) \). Is it merely an accident that \( K(\mathbb{Z}, 1) \) and \( K(\mathbb{Z}, 0) \) appear together here?

To answer this, we must make a short excursion into the topology of loop spaces and classifying spaces. In general, we can construct \( K(A, n+1) \) by taking the classifying space of \( K(A, n) \):

\[
\mathcal{B}K(A, n) = K(A, n+1)
\]

For example, \( \mathcal{B}K(A, 0) = \mathcal{BZ} = U(1) \) since

\[
0 \to H^0(X, \mathbb{Z}) \to H^0(X, \mathbb{R}) \to H^0(X, U(1)) \to H^1(X, \mathbb{Z}) \to 0
\]

is exact, so a \( \mathbb{Z} \)-bundle is given by counting the winding number of some \( U(1) \)-valued function. In other words, if \( \Sigma \to U(1) \) is the \( \mathbb{Z} \)-bundle

\[
\Sigma = \mathbb{R} \exp 2\pi i U(1)
\]

then every \( \mathbb{Z} \)-bundle \( E \to X \) satisfies

\[
E \cong f^*\Sigma
\]

for some \( f : X \to U(1) \).

Now let us apply the same trick to compute \( K(\mathbb{Z}, 2) \). \( K(\mathbb{Z}, 2) \) is the classifying space of \( K(\mathbb{Z}, 1) \cong U(1) \). Every \( U(1) \)-bundle is the pullback of the tautological bundle on some projective space \( \Sigma \to \mathbb{C}P^n \), so the classifying space is \( \mathbb{C}P^\infty \) (or the homotopically equivalent \( \mathbb{P}\mathcal{H} \)).

The above discussion presents the motto “\( H^n(X, \mathbb{Z}) \) classifies \( K(\mathbb{Z}, n) \)-bundles on \( X \)”. This gives us an easy way to find geometric realizations of \( H^n(X, \mathbb{Z}) \), but in practice these realizations become unwieldy very quickly. For example, the classifying space of \( PGL(\mathcal{H}) \)-bundles is a quotient of \( U(V_{HS}) \) where \( V_{HS} \) is the space of Hilbert-Schmidt operators on \( \mathcal{H} \).

All of this fits in to the more general program of realizing geometric structures like principal bundles on the loop space \( LX \) (or path space \( PX \) ) as somewhat more complex structures on the underlying space \( X \). In particular, we may take advantage of the fact that

\[
H^n(LX, A) \cong H^{n+1}(X, A)
\]

Even better, looping \( X \) is closely related to taking the classifying space of \( A \), so we have

\[
H^n(LX, A) \cong H^n(X, BA)
\]
In the case analyzed above, we can start with linebundles on $LX$ and find on the one hand

$$H^1(LX, U(1)) = H^1(X, BU(1)) = H^1(X, PGL(\mathcal{H}))$$

and on the other

$$H^1(LX, U(1)) = H^2(X, U(1)) = H^3(X, \mathbb{Z})$$

giving a purely abstract proof that $PGL(\mathcal{H})$-bundles on $X$ are classified by $H^3(X, \mathbb{Z})$. But this calculation also tells us why we might care — any time a linebundle appears naturally on loop space, there is a corresponding $PGL(\mathcal{H})$-bundle floating around on the base space.

Still, it would be nice to work out a construction of the projective Hilbert space bundle corresponding to a given class in $H^3(X, \mathbb{Z})$. It is extremely tempting to search for a family of classical mechanical systems over $X$ whose quantized space of states gives such a bundle. Perhaps the integral class could be read directly from the classical mechanics of the family.