The aim of this note is to define a strict 2-group associated to each group presentation and apply this 2-group to the problem of classifying arbitrary group extensions. Before we begin, we need to understand homomorphisms in the category of strict 2-groups.

Let $G$ and $G'$ be strict 2-groups (that is, categories internal to $\text{Groups}$). Then a 2-group homomorphism is a functor

$$\Phi : G \rightarrow G'$$

internal to $\text{Groups}$.

While this definition is conceptually useful, we will also need to understand $\Phi$ in terms of the underlying object- and arrow-homomorphisms. For the 2-group $G$, let $G_0$ be the group of objects, $G_1$ the group of arrows with source 1, $\alpha$ the action of $G_0$ on $G_1$ and $t : G_1 \rightarrow G_0$ the target map. $\Phi$ is determined by a pair of homomorphisms $(\varphi_0, \varphi_1)$ making the square

$$\begin{array}{ccc}
G_0 & \xrightarrow{\varphi_0} & G_0' \\
t & & t' \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{\varphi_1} & G_1'
\end{array}$$

commute.

$\Phi$ gives a homomorphism of the full arrow group $G_1 \ltimes_\alpha G_0$ only when the equation

$$\alpha'(\varphi_0(x))(\varphi_1(y)) = \varphi_1(\alpha(x)(y))$$

is satisfied. Conversely, any pair $(\varphi_0, \varphi_1)$ satisfying this equation and intertwining $t$ with $t'$ will give a homomorphism of 2-groups.
1 The Presentation 2-Group

Let \( \langle \Gamma \mid R \rangle \) be a presentation of the group \( G \). By a presentation we will mean that \( \Gamma \) is a set of symbols and \( R \) is a set of words in those symbols (or their inverses). For example, a presentation of \( SL(2, \mathbb{Z}) \) is given by

\[
\langle s, t \mid s^2, (st)^3 \rangle
\]

Recall the usual construction of a \( K(G, 1) \) from a presentation \( \langle \Gamma \mid R \rangle \): we start with a point, add a loop for each word in \( \Gamma \), add a 2-cell for each relation between words, a 3-cell for each syzygy between relations, and so forth. Note that this cell complex actually has nice algebraic structure: the 2-cells tell you how to get from one word to another, not merely that you can get from one word to another. This is precisely the sort of situation 2-groups are designed to deal with.

Let us now forget about the cells of dimension 3 and higher. Then the resulting cell complex is actually a 2-group:

**Definition 1.1.** Let \( \langle \Gamma \mid R \rangle \) be a presentation of \( G \). Then the associated **presentation 2-group** is the 2-group \( Pr_{\langle \Gamma \mid R \rangle} \) whose objects are the (reduced) words in \( \Gamma \). There is an arrow from \( w_1 \) to \( w_2 \) whenever \( w_2 = rw_1 \) for some product of relations \( r \).

The presentation 2-group can also be defined as a crossed module

\[
\bar{R} \xrightarrow{t} F_{\Gamma}
\]

where \( F_{\Gamma} \) is the free group on \( \Gamma \), \( \bar{R} \) is the normal closure of \( R \) in \( F_{\Gamma} \), and \( t \) is the inclusion map. The action of words on relations is simply

\[
\alpha(w)(r) = wrw^{-1}
\]

The intertwining property and Peiffer identity are easily verified.

Later, we will use \( Pr_G \) to denote the presentation 2-group associated to the “universal presentation” \( \langle G \mid \bar{R} \rangle \).

2 Group Cohomology

Let \( G \) be a group, \( A \) an abelian group, and \( \alpha : G \rightarrow Aut(A) \) an action of \( G \) on \( A \). We want to classify extensions

\[
1 \rightarrow A \rightarrow X \rightarrow G \rightarrow 1
\]

which are compatible with \( \alpha \).
We define the $n$-cochains $C^n(G, A)$ to be the abelian group of functions (not homomorphisms!) from $G^n$ to $A$. There is a coboundary operator $d : C^n(G, A) \longrightarrow C^{n+1}(G, A)$

\[
(df)(g_0, \ldots, g_n) = \alpha(g_0)(f(g_1, \ldots, g_n)) \\
+ \sum_{i=0}^{n-1} (-1)^i f(g_0, \ldots, g_i \cdot g_{i+1}, \ldots, g_n) \\
+ (-1)^n f(g_0, \ldots, g_{n-1})
\]

Direct computation shows that $d \circ d = 0$. As usual, we define $H^n(G, A)$ to be the kernel of $d$ (outgoing) modulo the image of $d$ (incoming).

For small values of $n$ we have

$H^0(G, A) = \{a \in A \mid \alpha(G)(a) = a\}$

$H^1(G, A) = \{f : G \longrightarrow A \mid f(xy) = f(x) + \alpha(x)(f(y)) \}$

$H^2(G, A) = \left\{ \frac{G^2 \longrightarrow A}{\ker d} \mid \alpha(x)(f(y, z)) - f(xy, z) + f(x, yz) - f(x, y) = 0 \right\}$

Now suppose an extension $A \longrightarrow X \xrightarrow{\pi} G$ compatible with $\alpha$ is given. Here, compatibility with $\alpha$ means that for each $x \in \pi^{-1}(g)$ and all $h$,

$x \cdot h \cdot x^{-1} = \alpha(g)(h)$

Choose an arbitrary section $\psi : G \longrightarrow X$ of $\pi$. We do not need $\psi$ to be a homomorphism, just an arbitrary map. To measure how badly $\psi$ fails to be a homomorphism, define the failure function

$f(x, y) = \psi(x) \cdot \psi(y) \cdot \psi(xy)^{-1}$

Note that $\pi(f(x, y)) = 1$, so $f$ actually takes values in $H$. In fact, $f$ is a cocycle:

\[
(df)(x, y, z) = \alpha(x)(f(y, z)) + f(xy, z)^{-1} + f(x, yz) + f(x, y)
\]

\[
+ \psi(xy) \cdot \psi(yz)^{-1} \cdot \psi(x)^{-1} + \psi(x) \cdot \psi(y) \cdot \psi(xy)^{-1}
\]

\[
= \alpha(x)(f(y, z)) + \phi(x) \cdot f(y, z) \cdot \phi(x)^{-1}
\]

\[
= 0
\]

********** signs are all messed up **********
3 Group Extensions

The goal of this section is to understand group extensions

\[
1 \rightarrow H \xrightarrow{i} X \xrightarrow{\pi} G \rightarrow 1
\]

of \(G\) by \(H\), where we do not assume \(G\) or \(H\) is abelian.

3.1 How 2-Groups Appear in the Extension Problem

First, we recall how split exact sequences lead to semidirect products — we will try to mimic this construction in the non-split case. Suppose we are given a splitting

\[
1 \rightarrow H \xrightarrow{i} X \xrightarrow{\pi} G \rightarrow 1
\]

so \(G \xrightarrow{s} X\) is a homomorphism with \(\pi \circ s = \text{id}\). \(s\) provides us with a “best” representative for each \(H\)-coset. We can then think of any element of \(X\) uniquely as a product of an element in \(G\) by an element in \(H\):

\[
(h, g) \mapsto h \cdot s(g) \in X
\]

The multiplication rule on \(H \times G\) must then be

\[
(h_2, g_2) \cdot (h_1, g_1) = h_2 \cdot s(g_2) \cdot h_1 \cdot s(g_1) = (h_2 \cdot s(g_2) \cdot h_1 \cdot s(g_2)^{-1}, g_2 g_1)
\]

Conversely, given any action \(G \xrightarrow{\alpha} \text{Aut}(H)\) we can construct a group \(X = G \rtimes_\alpha H\) which is a split extension of \(G\) by \(H\).

Unfortunately, most group extensions do not come from split extensions. On the other hand, we can always find a splitting \(G \xrightarrow{\psi} X\) if we do not require that \(\psi\) be a homomorphism.

\[
1 \rightarrow H \xrightarrow{i} X \xrightarrow{\psi} G \rightarrow 1
\]

A set function feels uncomfortable in the category of groups. Luckily, every set function \(G \xrightarrow{\Psi} X\) lifts to a unique homomorphism \(F_G \xrightarrow{\Psi} X\) where \(F_G\) is the free group on the elements of \(G\). Thus, conjugation by \(\Psi\) gives an action of \(F_G\) on \(H\). We will call this action \(\alpha : F_G \longrightarrow \text{Aut}(H)\). Explicitly,

\[
\alpha(w)(h) = \iota^{-1}(\Psi(w) \cdot \iota(h) \cdot \Psi(w)^{-1})
\]
If $r \in \tilde{R}$ then $\Psi(r) \in H$ since $\pi(\Psi(r)) = 1$ Thus, $\alpha(r)$ is an inner automorphism of $H$ when $r$ is a relation. This gives a nontrivial constraint on which homomorphisms $\Psi$ come from an extension: we must have a commutative diagram

\[
\begin{array}{c}
F_G & \xrightarrow{\alpha} & \text{Aut}(H) \\
\downarrow & & \downarrow \text{Ad} \\
\tilde{R} & \xrightarrow{\Psi} & H
\end{array}
\]

Essentially, the map $\Psi : \tilde{R} \to H$ measures how badly $\Psi$ fails to project to a homomorphism $\psi : G \to X$.

The point of this whole construction is that such a diagram is precisely a homomorphism of 2-groups $\Pr_G \xrightarrow{\varphi} \text{AUT}_H$ from the presentation 2-group of $G$ to the automorphism 2-group of $H$. Conversely, given such a homomorphism we can define an extension of $G$ by $H$ by reading this section backwards.

In summary, we have shown that

**Theorem 3.1.** Every extension of $G$ by $H$ induces a homomorphism of 2-groups

$$
\Pr_G \longrightarrow \text{AUT}_H
$$

and conversely, each homomorphism of 2-groups induces an extension of $G$ by $H$.

We can be even more explicit: let $\langle \Gamma_G \vert R_G \rangle$ be a presentation of $G$ and $\langle \Gamma_H \vert R_H \rangle$ a presentation of $H$. Then

$$
\langle \Gamma_G \cup \Gamma_H \mid R_H \cup R'_G \rangle
$$

is a presentation of $X$, where

$$
R'_G = \{ r \cdot \varphi_2(r)^{-1}, g \cdot h \cdot g^{-1} \cdot \varphi_1(g)(h)^{-1} \mid r \in \tilde{R}, g \in G, h \in H \}
$$

### 3.2 An Explicit Construction of the Group Product

In this section, we will find explicit formulas for the multiplication in $X$ in terms of the homomorphisms $\varphi_1$ and $\varphi_2$. Throughout this section, $\otimes$ will stand for concatenation in a free group.

Let us begin as above with a splitting of $H \to X \to G$ using a function $\psi$ (which is generally not a homomorphism). Let $\Psi$ be the lift of $\psi$ to $F_G$, so

$$
\Psi(g_1 \otimes \cdots \otimes g_k) = \psi(g_1) \cdot \psi(g_2) \cdots \psi(g_k)
$$
Then the homomorphisms $\varphi_i$ are defined by

$$\varphi_1(w)(h) = \Psi(w) \cdot h \cdot \Psi(w)^{-1}$$

and

$$\varphi_2(r) = \Psi(r)$$

where $w$ is an arbitrary word in $F_G$ and $r$ is a relation in $\bar{R}$. Note that $\varphi_2(r)$ is $H$-valued since it spells a relation in $G$ and therefore is in kernel of the quotient.

Now consider the set $H \times G$. We wish to understand how these pairs relate to elements of $X$. To each pair, we can associate the element

$$(h, g) \mapsto h \cdot \psi(g) \in X$$

and likewise, to each $x \in X$ we can associate the pair

$$x \mapsto (x \cdot \psi(\pi(x))^{-1}, \pi(x))$$

where $\pi$ is the quotient map. Now consider the multiplication in terms of these pairs:

$$(h_2, g_2) \cdot (h_1, g_1) = h_2 \cdot \psi(g_2) \cdot h_1 \cdot \psi(g_1)$$

$$= h_2 \cdot \varphi_1(g_2)(h_1) \cdot \psi(g_2) \cdot \psi(g_1)$$

$$= h_2 \cdot \varphi_1(g_2)(h_1) \cdot \Psi(g_2 \otimes g_1)$$

$$= h_2 \cdot \varphi_1(g_2)(h_1) \cdot \Psi(g_2 \otimes g_1 \otimes (g_2g_1)^{-1} \otimes g_2g_1)$$

$$= h_2 \cdot \varphi_1(g_2)(h_1) \cdot \varphi_2(g_2 \otimes g_1 \otimes (g_2g_1)^{-1}) \cdot \psi(g_2g_1))$$

$$= (h_2 \cdot \varphi_1(g_2)(h_1) \cdot \varphi_2(g_2 \otimes g_1 \otimes g_1^{-1}g_2^{-1}), g_2g_1)$$

We clearly see that the product in $X$ is just the semidirect product, twisted by $\varphi_2$.

Now we can easily describe generators and relations for $X$. If $\langle \Gamma_H | R_H \rangle$ is a presentation of $H$ and $\langle \Gamma_G | R_G \rangle$ a presentation of $G$, then $\langle \Gamma_H \cup \Gamma_G | R_X \rangle$ is a presentation of $X$, where $R_X$ contains the relations

$$g \otimes h = \varphi_1(g)(h) \otimes g$$

$$h_2 \otimes h_1 = h_2h_1$$

$$g_2 \otimes g_1 = \varphi_2(g_2 \otimes g_1 \otimes g_1^{-1}g_2^{-1}) \otimes g_2g_1$$

### 4 Classical Cases

In this section, we will suppose a homomorphism $\Pr_{G, \Phi} \longrightarrow \text{AUT}_H$ is given. $\Phi$ consists of a homomorphism on objects $\varphi_1 : F_G \longrightarrow \text{Aut}(H)$ and a homomorphism on arrows
\( \varphi_2 : \bar{R} \longrightarrow H \) with the appropriate intertwining properties:

\[
\begin{array}{ccc}
F_{G} & \overset{\varphi_1}{\longrightarrow} & \text{Aut}(H) \\
\downarrow s & & \downarrow \text{Ad} \\
\bar{R} & \overset{\varphi_2}{\longrightarrow} & H
\end{array}
\]

4.1 Direct and Semidirect Products

First, let us analyze what happens when \( \Phi \) is trivial for cells of various dimensions.

What happens if the homomorphism \( \Phi \) is “2-trivial” (meaning \( \varphi_2 \) takes everything to 1)? Then, as noted in the previous section, \( \varphi_1 \) projects to a homomorphism \( s : G \longrightarrow X \). In other words, \( \Phi \) is 2-trivial if and only if it induces a split extension of \( G \) by \( H \).

If \( \Phi \) is “1-trivial” (meaning \( \varphi_1 \) and \( \varphi_2 \) both take everything to 1), then \( G \) has a trivial action on \( H \) — the induced extension is simply the direct sum of \( G \) with \( H \).

It seems that direct products, semidirect products and general extensions are 0-, 1-, and 2-dimensional versions of the same phenomenon.

This also leads us to define a “co-semidirect product” which is trivial in dimension 1 but not in dimension 2. In terms of the underlying homomorphisms, \( \varphi_1 \) is trivial and \( \varphi_2 \) is not. This means that \( \varphi_2 \) must take values in the center \( Z(H) \) of \( H \).

********** what is going on with this?? **********

4.2 Relating \( \Phi \) to \( H^2(G, H) \)

If \( H \) is abelian, it is well-known that extensions of \( G \) by \( H \) with a given action \( G \overset{\alpha}{\longrightarrow} \text{Aut}(H) \) are classified by \( H^2(G, H) \). In this section, we will consider the connections between \( H^2(G, H) \) and the 2-group homomorphism \( \Phi \).

Since \( H \) is abelian, the map \( \text{Ad} : H \longrightarrow \text{Aut}(H) \) is trivial. This forces \( \varphi_1 \) to be trivial on \( i(\bar{R}) \). Equivalently, \( \varphi_1 \) descends to an action of \( G \) on \( H \). At the same time, all restrictions on \( \varphi_2 \) have been removed.

Note that every relation in \( \bar{R} \) can be written as a sequence of elementary relations

\[ r_{xy} = x \otimes y \otimes y^{-1}x^{-1} \]

Define the map \( f : G \times G \longrightarrow H \) by

\[ f(x, y) = \varphi_2(r_{xy}) = \varphi_2(x \otimes y \otimes y^{-1}x^{-1}) \]

\( f \) is a cocycle:

7
\[ df(x,y,z) = \varphi_1(x)(f(y,z)) - f(xy,z) + f(x,yz) - f(x,y) \]
\[ = \varphi_1(x)(f(y,z)) + \varphi_2(xyz \otimes z^{-1} \otimes y^{-1}x^{-1}) + \varphi_2(x \otimes yz \otimes z^{-1}y^{-1}x^{-1}) + \varphi_2(xy \otimes y^{-1} \otimes x^{-1}) \]
\[ = \varphi_1(x)(f(y,z)) + \varphi_2(x \otimes yz \otimes z^{-1} \otimes y^{-1} \otimes x^{-1}) \]
\[ = 0 \]

The last line follows from the fact that, since \( \Phi \) is a 2-group homomorphism,
\[ \varphi_1(g)(\varphi_2(r)) = \varphi_2(grg^{-1}) \]

This entire argument works just as well in reverse: if \( \alpha : G \to Aut(H) \) is an action and \( f : G \times G \to H \) a cocycle, then \( \varphi_1 \) is the extension of \( \alpha \) to \( F_G \) and
\[ \varphi_2(r_{xy}) = f(x,y) \]
Since \( r_{xy} \) generate the relations, this determines \( \varphi_2 \). To ensure consistency we must have
\[ \varphi_2(x \otimes y \otimes y^{-1}x^{-1}) \cdot \varphi_2(xy \otimes z \otimes z^{-1}y^{-1}x^{-1}) = \]
\[ \varphi_2(x \otimes y \otimes z \otimes z^{-1}y^{-1} \otimes x^{-1}) \cdot \varphi_2(x \otimes yz \otimes z^{-1}y^{-1}x^{-1}) \]
which is precisely the cocycle condition on \( f \).

### 4.3 Central Extensions of Loop Groups

### 5 The Problem of Equivalence

The problem of equivalence is that I can’t figure out equivalence.