The Étale Topology
a paean in honor of Grothendieck

John Hubbard
The story starts in 1949, with the Weil conjectures. Let \( X \subset \mathbb{P}^n(\bar{k}) \) be a smooth algebraic variety over \( \bar{k} = \mathbb{Z}/(p) \) and \( N_m \) the number of points of \( X \) over the field with \( p^m \) elements. Define

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\frac{d}{dt} \log Z_X(t) = \sum_{m=1}^{\infty} N_m t^{m-1}
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The story starts in 1949, with the Weil conjectures. Let $X \subset \mathbb{P}^n(\overline{k})$ be a smooth algebraic variety over $k = \mathbb{Z}/(p)$ and $N_m$ the number of points of $X$ over the field with $p^m$ elements. Define

$$\frac{d}{dt} \log Z_X(t) = \sum_{m=1}^{\infty} N_m t^{m-1}$$

the $\zeta$-function of $X$
What is a non-singular variety \( X \subset \mathbb{P}^n \) over \( k \)?

Let \( p_1, \ldots, p_m : k^{n+1} \to k \) be homogeneous polynomials.

Let \( \tilde{X} \subset \bar{k}^{n+1} \) be \( \tilde{X} = \{ x \mid p_1(x) = \cdots = p_m(x) = 0 \} \). Suppose at every nonzero \( x \) in \( \tilde{X} \) there exist \( 1 \leq i_1 < \cdots < i_l \leq m \) and a Zariski neighborhood \( U \) of \( x \) such that

\[
U \cap \tilde{X} = \{ x \in U \mid p_{i_1}(x) = \cdots = p_{i_l}(x) = 0 \}
\]

and

\[
\text{rank } \begin{bmatrix}
\frac{\partial p_{i_1}}{\partial x_0}(x) & \cdots & \frac{\partial p_{i_1}}{\partial x_n}(x) \\
\vdots & & \vdots \\
\frac{\partial p_{i_l}}{\partial x_0}(x) & \cdots & \frac{\partial p_{i_l}}{\partial x_n}(x)
\end{bmatrix} = l
\]

Then \( \bar{k}^* \) acts on \( \tilde{X}^* = \tilde{X} - \{0\} \), and \( X = \tilde{X}^*/\bar{k}^* \) is smooth of dimension \( n - l + 1 \).
Example: $X = \mathbb{P}^k$

For the field $\ell$ with $p^m$ elements, $\mathbb{P}^k(\ell)$ has

\[
\frac{p^{m(k+1)} - 1}{p^m - 1} = 1 + p^m + \cdots + p^{km}
\]

points. So

\[
\frac{d}{dt} \log Z_{\mathbb{P}^k}(t) = \sum_{m=1}^{\infty} \left(1 + p^m + \cdots + p^{km}\right) t^m
\]

\[
= \frac{1}{1 - t} + \frac{p}{1 - pt} + \cdots + \frac{p^k}{1 - p^k t}
\]

\[
= \frac{d}{dt} \log \frac{1}{(1 - t)(1 - pt) \cdots (1 - p^k t)}
\]

So

\[
Z_{\mathbb{P}^k}(t) = \frac{1}{(1 - t)(1 - pt) \cdots (1 - p^k t)}
\]
The Weil Conjectures

1. $Z_X(t)$ is a rational function:

$$Z_X(t) = \frac{p_1(t) \cdots p_{2k-1}(t)}{p_0(t) \cdots p_{2k}(t)}$$

where the $p_i$ are polynomials in $\mathbb{Z}[t]$, $\deg p_i = \beta_i$. 
The Weil Conjectures

1 \( \mathcal{Z}_X(t) \) is a rational function:

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2 \( \mathcal{Z}_X \left( \frac{1}{p^k t} \right) = \pm p^{\frac{k \chi}{2}} t^\chi \mathcal{Z}_X(t) \)

where \( \chi = \chi(X) = \sum_{i=0}^{2k} (-1)^i \beta_i \) is the Euler characteristic of \( X \).
The Weil Conjectures

1. $Z_X(t)$ is a rational function:

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where the $p_i$ are polynomials in $\mathbb{Z}[t]$, $\deg p_i = \beta_i$.

2. $Z_X\left(\frac{1}{p^kt}\right) = \pm p^{\frac{k\chi}{2}} t^\chi Z_X(t)$

where $\chi = \chi(X) = \sum_{i=0}^{2k} (-1)^i \beta_i$ is the Euler characteristic of $X$.

3. $p_i(t) = \prod_{j=1}^{\beta_i}(1 - \alpha_{ij}t)$

where the $\alpha_{ij}$ are algebraic integers and $|a_{ij}| = p^{i/2}$. 
Let \( X \) be a smooth compact complex manifold of dimension \( k \), and \( \varphi : X \to X \) an endomorphism. Let \( N_m(\varphi) = \# \text{Fix}(\varphi^m) \) which we will suppose is finite. Define

\[
\frac{d}{dt} \log Z_\varphi(t) = \sum_{m=1}^{\infty} N_m(\varphi)t^{m-1}
\]
Let \( \varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) be a polynomial of degree \( d \). Then

\[
N_m(\varphi) = d^m + 1
\]

(including the fixed point at infinity), and

\[
\frac{d}{dt} Z_\varphi(t) = \sum_{m=1}^{\infty} (d^m + 1)t^{m-1}
\]

\[
= \frac{d}{dt} \log \frac{1}{(1-t)(1-dt)}
\]

So \( Z_\varphi(t) = \frac{1}{(1-t)(1-dt)} \). Note that

\[
Z_\varphi \left( \frac{1}{dt} \right) = \frac{1}{(1 - \frac{1}{dt}) \left( 1 - \frac{d}{dt} \right)} = \frac{dt^2}{(dt-1)(t-1)}
\]

\[
= d^\frac{1}{2} t^\chi Z_\varphi(t)
\]

since \( \chi = \chi(\mathbb{P}^1) = 2 \).
Now let $X$ be the square torus $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$, $\varphi : X \to X$ induced by $z \mapsto dz$. Then $\# \text{Fix}(\varphi^m) = (d^m - 1)^2$, so

\[
\frac{d}{dt} Z_\varphi(t) = \sum_{m=1}^{\infty} (d^m - 1)^2 t^{m-1}
\]

\[
= \sum d^{2m} t^{m-1} - 2 \sum d^m t^{m-1} + \sum t^{m-1}
\]

\[
= \frac{d}{dt} \log \frac{(1 - dt)^2}{(1 - d^2t)(1 - t)}
\]

So we get

\[
Z_\varphi(t) = \frac{(1 - dt)^2}{(1 - d^2t)(1 - t)}.
\]
Example 2, cont’d.

From \( Z_\varphi(t) = \frac{(1 - dt)^2}{(1 - d^2t)(1 - t)} \), we may compute

\[
Z_\varphi \left( \frac{1}{d^2t} \right) = \frac{\left(1 - \frac{\partial}{\partial^2 t}\right)^2}{\left(1 - \frac{\partial^2}{\partial^2 t}\right)(1 - \frac{1}{d^2 t})}
\]

\[
= \frac{(dt - 1)^2}{(t - 1)(d^2t - 1)} \cdot \frac{d^2 t \cdot t}{d^2 t^2}
\]

\[= t^\chi (d^2) \frac{\chi}{2} Z_\varphi(t) \]

since \( \chi = \chi(X) = 0 \).
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Why do these formulas come out so nicely?
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Why do these formulas come out so nicely?

*The Lefschetz fix point formula and Poincaré duality!*
Applying the Lefschetz fix point formula

\[
\# \text{Fix } \varphi^m = \sum_{i=0}^{2k} (-1)^i \text{tr} \left( \varphi^* : H^i(X) \to H^i(X) \right) = \sum_{i=0}^{2k} (-1)^i \sum_{j=1}^{\beta_i} \lambda_{i,j}^m
\]

where \( \lambda_{i,j} \) are the eigenvalues of \( \varphi^* : H^i(X) \to H^i(X) \). So

\[
\frac{d}{dt} \log Z_\varphi(t) = \sum_{m=1}^{\infty} \sum_{i=0}^{2k} (-1)^i \sum_{j=1}^{\beta_i} \lambda_{i,j}^m t^{m-1}
\]

\[
= \sum_{i=0}^{2k} (-1)^i \sum_{j=1}^{\beta_i} \frac{\lambda_{ij}}{1 - \lambda_{ij} t} = \frac{d}{dt} \log \frac{\prod_{i \text{ odd}} \prod_j (1 - \lambda_{ij})}{\prod_{i \text{ even}} \prod_j (1 - \lambda_{ij} t)}
\]

Thus \( Z_\varphi(t) = \frac{p_1(t) \cdots p_{2k-1}(t)}{p_0(t) \cdots p_{2k}(t)} \) where \( p_i \) is defined by

\( p_i(t) = \det \left( \text{id} - t(\varphi^* : H^i(X) \to H^i(X)) \right) \). In particular, \( p_i \) is an integral polynomial of degree \( \beta_i \).
Applying Poincaré Duality

Moreover, Poincaré duality says:

If $\lambda_{ij}$ is an eigenvalue of $\varphi^*: H^i(X) \to H^i(X)$, then $\frac{d}{\lambda_{ij}}$ is an eigenvalue of $\varphi^*: H^{2k-i}(X) \to H^{2k-i}(X)$

This leads to

$$p_i\left(\frac{1}{dt}\right) = \frac{\det(\varphi^*: H^i(X) \to H^i(X))}{(dt)^\beta_i} p_{2k-i}(t)$$
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and

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and

\[
\det(\phi^* : H^i(X) \to H^i(X)) \cdot \det(\phi^* : H^{2k-i}(X) \to H^{2k-i}(X)) = d^{\beta_i}
\]

Finally,

\[
Z\left(\frac{1}{dt}\right) = \frac{p_1\left(\frac{1}{dt}\right) \cdots p_{2k-1}\left(\frac{1}{dt}\right)}{p_0\left(\frac{1}{dt}\right) \cdots p_{2k}\left(\frac{1}{dt}\right)} = \pm t^{\chi} d^{\frac{\chi}{2}} Z(t)
\]
What does this have to do with counting points of varieties over finite fields?

The Frobenius map \( \phi: x \mapsto x^p \) is linear in characteristic \( p \):

\[(a + b)^p = a^p + b^p\]

It follows that if \( X \) is the set of points in \( \mathbb{P}^n(\mathbb{Z}/p\mathbb{Z}) \) satisfying some collection of homogeneous equations with coefficients in \( \mathbb{Z}/p\mathbb{Z} \), then \( \phi: X \to X \).
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\ldots

the points $X_m$ of $X$ with coordinates in $\mathbb{F}_{p^m}$ are $\text{Fix}(\varphi^m)$
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\vdots

the points $X_m$ of $X$ with coordinates in $\mathbb{F}_{p^m}$ are $\text{Fix}(\varphi^m)$

So if there is a cohomology for varieties over a finite field for which Lefschetz fixed point theorem and Poincaré duality hold, they can be used to prove the Weil conjectures.
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Answer: NO!

Serre (CRAS 258(1964) 4194 – 4195) shows that there exists a surface defined by equations with coefficients in $\mathbb{K} = \mathbb{Q}(\xi)$, $\xi^{23} = 1$, such that for two embeddings $\mathbb{K} \hookrightarrow \mathbb{C}$, the complex varieties have different fundamental groups.

Does this mean that the task of defining an appropriate cohomology theory is hopeless?

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Does this mean that the task of defining an appropriate cohomology theory is hopeless?

Answer: NO! Grothendieck did it.

One central ingredient: Čech covers.
Let $X$ be a topological space, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of $X$.

The nerve $N(\mathcal{U})$ is the simplicial complex whose vertices are elements of $I$, and where $(i_0, \ldots, i_k)$ span a simplex iff $U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset$.

**Theorem (Čech)**

*If the $U_i$ are all contractible, as well as all intersections $U_{i_0} \cap \cdots \cap U_{i_k}$, then $N(\mathcal{U})$ has the same homotopy type as $X$.***
It doesn’t seem that this can be applied in our algebraic setting.

**Example**

If $X = \mathbb{P}^1(\mathbb{C})$ then the Zariski open sets are the complements of finite sets. So the nerve of every open cover is a simplex, which is always contractible. This actually applies to all irreducible varieties over $\mathbb{C}$. 
Čech Covers

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If $X = \mathbb{P}^1(\mathbb{C})$ then the Zariski open sets are the complements of finite sets. So the nerve of every open cover is a simplex, which is always contractible. This actually applies to all irreducible varieties over $\mathbb{C}$.

Grothendieck’s answer: we have the wrong open sets!
Redefine: “open set of $X$” $\rightsquigarrow$ algebraic variety $U$, together with a local isomorphism $\varphi : U \to X$. 
The étale topology

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- **Redefine**: “open cover” $\rightsquigarrow$

\[
\mathcal{U} = \left\{ (U_i, \varphi_i)_{i \in I} \mid \bigcup_{i \in I} \varphi_i(U_i) = X \right\}
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- **Redefine:** “Nerve of \( \mathcal{U} \)”:
  - vertices \( \leadsto \) elements of \( I \)
  - \( k \)-simplices \( \leadsto \) components of \( U_{i_0} \times X U_{i_1} \times X \cdots \times X U_{i_k} \)
  
  (this gives a semi-simplicial complex: there may be several edges joining vertices, etc.)
A simple étale cover

Example: A cover consisting of a single open set.

\[ \varphi \]

In ordinary topology this gives nothing new: every étale cover has a standard refinement.
In all cases, we will consider

\[ \lim_{\mathcal{U} \text{ cover of } X} \mathcal{N}(\mathcal{U}) = \lim_{\mathcal{U} \text{ étale cover of } X} \mathcal{N}(\mathcal{U}) = X \]

We can restate: if \( X \) is locally contractible, then

\( X \) has the same homotopy type
In all cases, we will consider

$$\lim_{\mathcal{U} \text{ cover of } X} \mathcal{N}(\mathcal{U})$$

We can restate: if $X$ is locally contractible, then

$$\lim_{\mathcal{U} \text{ cover of } X} \mathcal{N}(\mathcal{U}) = \lim_{\mathcal{U} \text{ étale cover of } X} \mathcal{N}(\mathcal{U}) = X$$

has the same homotopy type

For an algebraic variety with the Zariski topology, the situation is completely different!
We can’t of course get étale covers by contractible sets. What we do get is the following:

For any affine variety $U$ over $\mathbb{C}$ and every $\alpha \in H^j(U, \mathbb{Z}/m)$ there exists an étale cover $\mathcal{V} = (V_i, \varphi_i)$ of $U$ such that $\varphi^* \alpha = 0$ for all $i$. 

Example $U = \mathbb{C} - \{0\}$, $\alpha$ the nonzero element of $H^1(U, \mathbb{Z}/2)$. Then $\mathcal{V} = (\mathbb{C} - \{0\}, z \mapsto z^2)$ and $\varphi^* \alpha = 2\alpha = 0$ in $H^1(\mathbb{C} - \{0\}, \mathbb{Z}/2)$. 

John Hubbard The Étale Topology
The étale topology

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Example

\( U = \mathbb{C} - \{0\} \), \( \alpha \) the nonzero element of \( H^1(U, \mathbb{Z}/2) \). Then

\[
\mathcal{V} = (\mathbb{C} - \{0\}, z \mapsto z^2)
\]

and

\( \varphi^* \alpha = 2\alpha = 0 \)

in \( H^1(\mathbb{C} - \{0\}, \mathbb{Z}/2) \).
Profinite spaces

Every cohomology class \((\text{with finite coefficients})\) arising in an étale cover can be killed by passing to a finite cover.

To state clearly what this gives, we need the notion of a profinite space.

A **group** is **profinite** if the natural map

\[
G \to \lim_{\substack{\text{H of finite index in } G \to G/H}}
\]

is an isomorphism.
A space is profinite if the homotopy group $\pi_i(X)$ is profinite for all $i$.

The profinite completion $\hat{X}$ of a space $X$ is a profinite space with a map $X \to \hat{X}$, such that for every map $f : X \to Y$ with $Y$ profinite, there exists a unique $\hat{f} : \hat{X} \to Y$ such that the diagram

$$
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{f}} & Y \\
\downarrow \quad \quad \quad \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

commutes.
Finally we get Grothendieck’s answer.

**Theorem**

If $X$ is a smooth variety over $\mathbb{C}$ then the natural map

$$X_{\text{trans}} \longrightarrow \lim_{\mathcal{U} \text{ étale cover of } X} \mathcal{N}(\mathcal{U})$$

is the profinite completion of $X$.

Thus, the étale topology “sees” everything that can be see with finite coefficients.
This focuses attention on:

What is $X_{\text{ét}}$ for other schemes?

The first one that comes to mind is: what is $(\text{Spec }\mathbb{Z})_{\text{ét}}$?

The answer is *simply amazing*. 
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  - every algebraic extension of $\mathbb{Q}$ is ramified somewhere

- $(\text{Spec } \mathbb{Z}/p)_{\text{ét}}$ “is” a circle embedded in $(\text{Spec } \mathbb{Z})_{\text{ét}}$, i.e., a knot.
If $X$ is a smooth variety over $\mathbb{Z}/p$, it is true that

$$X_{\text{ét}} = \lim_{\leftarrow} \mathcal{N}(U)$$

$U$ étale cover of $X$

satisfies Poincaré duality (in dimension $2 \dim X$) and the Lefshetz fixed point theorem so long as one uses finite coefficients $\mathbb{Z}/\ell$, or more generally profinite coefficients $\mathbb{Z}_\ell$, the $\ell$-adic integers.

This allowed Grothendieck (assisted at the end by Deligne) to prove the Weil conjectures.
Three amazing links

This knot has an Alexander polynomial, that is non-trivial (its constant term $\neq 1$) if $p$ is irregular. These knots can link: the linking number is given by the Hilbert symbol.

Thus we see amazing links: