Related Rates Solutions

July 10, 2007

1. (a) **What we know:** Let’s call the distance along the ground from the truck to the pulley $T(t)$, the distance from the truck’s bumper to the pulley $D(t)$, and the height of the weight off of the ground $h(t)$. The problem tells us

\[
\frac{dT}{dt} = 2.5 \text{ ft/s}
\]

From the Pythagorean theorem and the fact that the bumper is 2 feet off the ground, we know

\[
D^2 = T^2 + 18^2
\]

and the fact that the rope isn’t stretching or changing length says that

\[
D + (20 - h) = \text{const}
\]

(b) **What we want:** The problem is asking how fast the weight is rising when the truck is 15 feet away ($T(t) = 15$), so we want to know $dh/dt$.

(c) **Wish List:** If we knew $dh/dT$ we could get what we want since

\[
\frac{dh}{dT} \cdot \frac{dT}{dt} = \frac{dh}{dt}
\]

(d) **Getting what we want:** If we use implicit differentiation on the two algebraic equations that we found, we get

\[
2D \frac{dD}{dt} = 2T \frac{dT}{dt}
\]

and

\[
\frac{dD}{dt} = \frac{dh}{dt} = 0
\]

The second equation means that $dh = dD$. Using this in the first equation, we can solve for $dh/dT$ (what we wanted):

\[
2D \frac{dh}{dt} = 2T \frac{dT}{dt}
\]
so
\[ \frac{dh}{dT} = \frac{T}{D} \]

This means that
\[ \frac{dh}{dt} = \frac{dh}{dT} \cdot \frac{dT}{dt} = \frac{T}{D} \cdot 2.5 \text{ ft/s} \]

We are interested in what happens when \( T = 15 \):
\[ \frac{dh}{dt} = 15 \cdot 2.5 \text{ ft/s} \]

When \( T = 15 \), the Pythagorean theorem tells us that \( D = \sqrt{15^2 + 18^2} \), so altogether we have
\[ \frac{dh}{dt} = \frac{15 \cdot 2.5}{\sqrt{15^2 + 18^2}} \text{ ft/s} \]

which is about 1.6 feet per second.

2. (a) **What we know:** Let’s call the angle between the hands \( \theta(t) \) and the distance between the tips \( D(t) \). The law of cosines tells us that
\[ D^2 = 2^2 + 3^2 + 2 \cdot 2 \cdot 3 \cdot \cos(\theta) \]

since the hour hand has length 2 and the minute hand has length 3. Since the hour hand goes around the clock once per 12 hours, it must move at a rate of \( 360/12 = 30 \) degrees per hour, or 0.5 degrees per minute. Since the minute hand moves once around the clock per hour, it must move at a rate of 360 degrees per hour, or 6 degrees per minute. Altogether, this means that the angle between them changes like
\[ \frac{d\theta}{dt} = 5.5 \text{ degrees/min} = \frac{2\pi \cdot 5.5}{360} \text{ rad/min} = \frac{11\pi}{360} \text{ rad/min} \]

so
\[ \theta(t) = \frac{11\pi}{360} t \text{ min} \]

(b) **What we want:** We want to how fast the distance between the tips of the hands is changing: \( dD/dt \).

(c) **Wish List:** If we knew \( dD/d\theta \) we could get what we want by:
\[ \frac{dD}{dt} = \frac{dD}{d\theta} \cdot \frac{d\theta}{dt} \]

(d) **How to get what we want:** If we differentiate the law of cosines, we get
\[ 2D \, dD = -12 \sin(\theta) d\theta \]
so 

\[
\frac{dD}{d\theta} = -6 \frac{\sin(\theta)}{D}
\]

Now we can solve for \(dD/dt\):

\[
\frac{dD}{dt} = \frac{dD}{d\theta} \cdot \frac{d\theta}{dt} = -6 \frac{\sin(\theta)}{D} \cdot \frac{11\pi}{360} = -\frac{11\pi \sin(\theta)}{D}
\]

Now all we need to know is \(\theta\) and \(D\) at 2:30. 2:30 is 150 minutes after noon, so \(t = 150\). This means that

\[
\theta(150) = \frac{-11 \cdot \pi \cdot 150}{360} = \frac{-55 \pi}{12}
\]

and using the law of cosines equation,

\[
D(150) = \sqrt{13 + 12 \cos\left(\frac{-55 \pi}{12}\right)}
\]

Plugging these in finally gives us the answer

\[
\frac{dD}{dt} = \frac{-11\pi \sin\left(\frac{-55 \pi}{12}\right)}{\sqrt{13 + 12 \cos\left(\frac{-55 \pi}{12}\right)}}
\]

3. (a) What we know: The surface area of the reservoir is given by \(\pi r^2\) and the volume by

\[
V = \frac{1}{3} \pi r^2 h
\]

The problem tells us that the volume decreases by twice the surface area per week:

\[
\frac{dV}{dt} = 2\pi r^2 \text{ m}^3/\text{wk}
\]

We know that the reservoir is shaped like a right circular cone, so we also have

\[
h = r
\]

since the cross-section is a 45 degree right triangle.

(b) What we want: We need to find how fast the height of the water is decreasing: \(dh/dt\)

(c) Wish List: If we knew \(dh/dV\) then we could get what we want by

\[
\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}
\]

(d) How to get what we want: The second equation means that we can replace each \(r\) with an \(h\) in the first equation, giving

\[
V = \frac{1}{3} \pi h^3
\]

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Differentiate this to get
\[ dV = \pi h^2 dh \]
so
\[ \frac{dh}{dV} = \frac{1}{\pi h^2} \]
Now we can get what we want:
\[ \frac{dh}{dt} = \frac{1}{\pi h^2} \cdot 2\pi r^2 \text{ m}^3/\text{wk} = 2 \text{ m}^3/\text{wk} \]
where we used \( r = h \) again.

4. (a) I’ll approach this problem a little more directly than the others. We should probably work in feet, so the train is moving at 528000 feet per hour. We will call the position of the train \( T(t) \), so
\[ \frac{dT}{dt} = 528000 \text{ ft/hr} \]
Let us figure out how fast the wheels are spinning. Since the inner wheel is the one driving the train and the inner wheel has a radius of 1.3 feet, the train moves forward \( 2\pi \cdot 1.3 \) feet per revolution. Therefore, the wheels make
\[ \frac{528000}{2\pi \cdot 1.3} \text{ revolutions/hr} \]
If \( \theta(t) \) is the angle that the wheel has turned, this says that
\[ \frac{d\theta}{dt} = \frac{528000}{2\pi \cdot 1.3} \text{ revs/hr} \]
A point on the bottom of the outer rim \( B \) is moving backwards at a rate of
\[ \frac{dB}{dt} = -2\pi \cdot 1.5 \cdot \frac{d\theta}{dt} \text{ ft/hr} \]
so the point on the bottom of the outer rim is moving backwards at the rate
\[ \frac{dB}{dt} = \frac{-2\pi \cdot 1.5 \cdot 528000}{2\pi \cdot 1.3} \text{ revs/hr} = -528000 \cdot \frac{1.5}{1.3} \text{ revs/hr} \]
relative to the axle of the train. This means that relative to the ground, the point is moving with velocity
\[ \frac{dB}{dt} + \frac{dT}{dt} = 528000(-\frac{1.5}{1.3} + 1) \text{ ft/hr} = 100 \cdot (1 - \frac{1.5}{1.3}) \text{ mi/hr} \]
or about 15.4 miles per hour backwards!