1 Week 1

1.1 Nilsquare Infinitessimals

Remember how in algebra you needed to add a special “number” $i$ with

$$i^2 = -1$$

in order to make your equations all have nice solutions? In calculus, we also need to enrich our number system by adding new (but strange!) “numbers” in order to capture our intuitive ideas about what “infinitely close to” should mean.

The new numbers we add are called infinitessimals and are infinitely small. The first kind of infinitesimal we will use is called a “nilsquare infinitesimal”. This is a special new kind of number $dx$ such that

$$dx > 0$$

but

$$(dx)^2 = 0$$

dx is so small that its square is nothing at all!

Let us do some computations with $dx$ for practice.

*Example.* We will compute $(5 + 3dx)^2$. Expanding the product gives

$$(5 + 3dx)^2 = 25 + 2 \cdot 5 \cdot 3dx + 9(dx)^2 = 25 + 30dx$$

using the fact that $(dx)^2$ is zero.
Example. It is true (but we won’t be able to prove until later) that $e^{dx} = 1 + dx$. What is $e^{2dx}$? We could compute this two ways, depending on which exponential rules we like most. Here is one way:

$$e^{2dx} = e^{dx} \cdot e^{dx} = (1 + dx) \cdot (1 + dx) = 1 + 2dx + (dx)^2 = 1 + 2dx$$

Here is another way:

$$e^{2dx} = (e^{dx})^2 = (1 + dx)^2 = 1 + 2dx + (dx)^2 = 1 + 2dx$$

The point of adding these infinitesimals is that we can think of the points $x + dx$ and $x$ as “neighbors”, even though on the number line no two points are literally adjacent to each other. This lets us define the derivative of a function by the equation

$$f'(x) = \frac{f(x + dx) - f(x)}{(x + dx) - x} = \frac{f(x + dx) - f(x)}{dx}$$

For example, let us prove that the derivative of $y(x) = x^2$ is $y'(x) = 2x$. Using the definition of the derivative,

$$y'(x) = \frac{y(x + dx) - y(x)}{dx} = \frac{1}{dx}(x^2 + 2x\, dx + (dx)^2 - x^2) = \frac{1}{dx}(2x\, dx) = 2x$$

(remember that $(dx)^2 = 0$!)

### 1.2 The Differential

Suppose we have a function $f$. If we plug in a real number like 5, we will get out a real number. But if we plug in a number with an infinitesimal part like $5 + dx$, we will get out a number which also has an infinitesimal part. Using the example of $y(x) = x^2$ from above,

$$y(x + dx) = (x + dx)^2 = x^2 + 2x\, dx = y(x) + y'(x)\, dx$$

The infinitesimal part of $y(x + dx)$ is called the differential of $y$ and is denoted $dy$. With this notation, we can write

$$y(x + dx) = y(x) + dy$$
Let us compute the differential of \( f(x) = \sin(x) \) using the yet-to-be proved facts that \( \sin(dx) = dx \) and \( \cos(dx) = 1 \). We will also need the angle-sum formula

\[
\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)
\]

from trigonometry. Now, to compute the differential \( df \):

\[
f(x + dx) = \sin(x + dx) = \sin(x) \cos(dx) + \cos(x) \sin(dx) = \sin(x) \cdot 1 + \cos(x) \cdot dx = \sin(x) + \cos(x) \ dx
\]

Since the infinitessimal parts must be equal, we have shown that when \( f(x) = \sin(x) \),

\[
df = \cos(x) \ dx
\]

We could also state this as

\[
d(sin x) = \cos x \ dx
\]

Here is a step-by-step method for computing the differential of a function \( y(x) \).

1. Plug \( x + dx \) into \( y \). In other words, compute \( y(x + dx) \).

2. Manipulate the result algebraically, using the fact that \( dx^2 = 0 \). You will now have a sum of several terms. Some of these terms are finite and others are infinitessimal. The infinitessimal terms are the ones multiplied by \( dx \).

3. Collect the finite terms with each other, and the infinitessimal terms with each other. Factor out a \( dx \) from the infinitessimal terms.

4. The part of the expression involving \( dx \) is the differential \( dy \) which we were after.

Let us apply this method to the function \( y(x) = x^2 - 3x \).
1. First, we plug in $x + dx$ and get

$$y(x + dx) = (x + dx)^2 - 3(x + dx)$$

2. Now, we expand this expression algebraically to get

$$x^2 + 2x \, dx + dx^2 - 3x - 3dx$$

Since $dx$ is infinitesimally small, $dx^2 = 0$ leaving us with

$$x^2 + 2x \, dx - 3x - 3dx$$

3. Collect the finite terms and the infinitesimal terms:

$$\underbrace{x^2 - 3x + 2x \, dx - 3dx}_{\text{finite}} + \underbrace{dx^2}_{\text{infinitesimal}}$$

We can factor a $dx$ out of the infinitesimal part to get

$$\underbrace{x^2 - 3x + (2x - 3)\cdot dx}_{\text{finite}} + \overbrace{dx}^{\text{infinitesimal}}$$

4. The infinitesimal part of that expression is $dy$, so we have shown that

$$dy = (2x - 3) \, dx$$

Let us finish by applying these steps to the long example from class. We will compute the differential $dy$ where of $y(x) = xe^x$ by using the fact (to be proved in the future!) that $e^{dx} = 1 + dx$. We will also need the exponential identity $e^{a+b} = e^a \cdot e^b$.

1. First, we plug in $x + dx$ and get

$$y(x + dx) = (x + dx)e^{x + dx}$$

2. Now, we manipulate this expression algebraically to get

$$(x + dx)e^x e^{dx}$$

Since $e^{dx} = 1 + dx$, this is equal to

$$(x + dx) \cdot e^x \cdot (1 + dx)$$
Multiplying this out gives us
\[ e^x \cdot (x \cdot 1 + dx \cdot 1 + x \cdot dx + dx \cdot dx) \]

We know that \( dx^2 = 0 \) leaving us with
\[ e^x \cdot (x + dx + x \cdot dx) = xe^x + e^x \, dx + xe^x \, dx \]

3. Collect the finite terms and the infinitesimal terms:
\[ xe^x + e^x \, dx + xe^x \, dx \]

We can factor a \( dx \) out of the infinitesimal part to get
\[ xe^x + (e^x + xe^x) \cdot dx \]

4. The infinitesimal part of that expression is \( dy \), so we have shown that
\[ dy = (e^x + xe^x) \, dx \]

The moral is that by following this algorithm and being clever in our algebraic manipulations, we can compute the differential of most nice functions in a precise, step-by-step way.

1.3 The Differential and the Derivative

The differential \( dy \) and the derivative \( y' \) are very closely related notions. Let us explore the exact relationship between them. Remember that the definition of the derivative \( y'(x) \) of \( y \) is
\[ y'(x) = \frac{y(x + dx) - y(x)}{dx} \]

and the differential \( dy \) is the infinitesimal part of \( y(x + dx) \). In formulas, this means that
\[ y(x + dx) = y(x) + dy \]

What happens if we plug this second equation into the first one?
\[ y'(x) = \frac{y(x) + dy - y(x)}{dx} = \frac{dy}{dx} \]
In other words, the derivative $y'$ is just the differential divided by $dx$. This also gives us a new way of saying “the derivative of $y$”: we could either write this as

$$y'$$

or as

$$\frac{dy}{dx}$$

In the last section, we saw that the differential of $x^2$ was $2x\,dx$. This means that the derivative is just

$$(x^2)' = \frac{2x \, dx}{dx} = 2x$$

In the case of $\sin(x)$, the differential was $\cos(x)\,dx$ and so the derivative is

$$(\sin x)' = \frac{\cos(x) \, dx}{dx} = \cos x$$

Likewise for the more complicated example of $xe^x$: the differential was $(e^x + xe^x)\,dx$, so the derivative is just

$$(xe^x)' = \frac{(e^x + xe^x) \, dx}{dx} = e^x + xe^x$$

So to find the derivative of a function, just compute its differential and divide by $dx$!

Of course, this all works in the other direction as well: since

$$\frac{dy}{dx} = y'(x)$$

multiplying on both sides by $dx$ gives us the equation

$$dy = y'(x)dx$$

So to find the differential of a function, just compute its derivative and multiply by $dx$!
1.4 Exercises

1. Show by direct computation that

\[ \frac{1}{1 + dx} = 1 - dx \]

(Hint: what happens when you multiply both sides by \(1 + dx\)?)

2. If \(y(3 + dx) = 7 + 2dx\), what is \(y'(x)\)? What is \(y(x)\)? What is \(y(x+5dx)\)?
A hint for this last question: think about what \(y(x+5dx) - y(x)\) means geometrically.

3. Use the angle sum formula

\[ \cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \]

to compute the differential of \(\cos(x)\). You will need to use the facts that \(\cos(dx) = 1\) and \(\sin(dx) = dx\).

4. Find the derivative of \(f(x) = x^2 + 3x - 7\) using infinitessimals.

5. Suppose that \(f(x) = 7, \ g(x) = 1, \ f'(x) = 3\) and \(g'(x) = -1\). What is \(df\)? \(dg\)? Compute the value of \(f(x + dx) \cdot g(x + dx)\)

Having trouble? Email me any and all questions: noonan@math.cornell.edu
1.5 Continuity

What does it mean to say that a function is continuous? One intuitive definition goes like this:

\[ f(x) \text{ is continuous if you can trace along its graph without lifting your pen.} \]

This is OK as far as intuition goes, but it does not really help us to prove that a certain function is continuous, nor does it help very much to give us new insight into continuity. So let us think about this definition a little more deeply.

What does it mean to say that you can draw the graph without lifting your pen? One way to formalize this is to say that whenever we move our pen a tiny amount

\[ dx \]

in the \( x \)-direction, we only need to move our pen a tiny amount

\[ dy \]

in the \( y \)-direction to keep tracing along the graph\(^1\).

We can state this fact even more precisely if we add a new phrase to our language of calculus. Let us write

\[ x \approx y \]

and say “\( x \) is infinitessimally close to \( y \)” if the finite parts of \( x \) and \( y \) are equal. For example,

\[ 7 + 2dx \approx 7 - 5dx \]

but

\[ 7 + 2dx \not\approx 6 + 2dx \]

Saying that \( f(x) \) is continuous is the same as saying that when \( x \) is very close to \( y \), \( f(x) \) is very close to \( f(y) \). Using our new term \( \approx \), we can make the definition

\[ \text{Saying “} f(x) \text{ is continuous” means that if } x \approx y, \text{ then } f(x) \approx f(y). \]

\(^1\)So the derivative \( dy/dx \) is the direction our pen is moving in!
Let us use this definition to show that \( f(x) = x^2 \) is a continuous function. First of all, if \( x \approx y \) then it must be true that

\[
y = x + dz
\]

for some infinitesimal \( dz \). So let us plug both \( x \) and \( y \) into \( f \) and see what we find.

\[
f(x) = \underbrace{x^2}_{\text{finite}}
\]

\[
f(y) = f(x + dz) = (x + dz)^2 = \underbrace{x^2}_{\text{finite}} + 2x dz \underbrace{dz}_{\text{infinitesimal}}
\]

Notice that \( f(x) \) and \( f(y) \) have the same finite parts, so \( f(x) \approx f(y) \). According to our definition, this means that \( f \) is a continuous function.

Here is a step-by-step method for seeing if a function \( f \) is continuous at a point \( a \):

1. Compute \( f(a) \). This checks the value of \( f \) at \( a \).
2. Compute \( f(a + dx) \). This checks the value of \( f \) a little to the right of \( a \).
3. Compute \( f(a - dx) \). This checks the value of \( f \) a little to the left of \( a \).
4. If they all have the same finite part then we say “\( f \) is continuous at \( a \)”. If any of them have different finite parts, we say “\( f \) is discontinuous at \( a \)” or “\( f \) has a discontinuity at \( a \)”.

Now think about the function \( s(x) \) which is \( 3x \) when \( x \leq 0 \) and \( (x + 1)^2 \) when \( x > 0 \). Is \( s \) continuous at 0? To find out, let us apply the step-by-step method with \( a = 0 \):

1. \( s(a) = s(0) = 3 \cdot 0 = 0 \) since \( a \leq 0 \).
2. \( s(a + dx) = s(dx) = (dx + 1)^2 = 1 + 2dx \) since \( a + dx > 0 \).
3. \( s(a - dx) = s(-dx) = -3 \cdot dx \) since \( a - dx < 0 \).
4. The finite parts of \( s(a) \) and \( s(a - dx) \) are both 0 but the finite part of \( f(a + dx) \) is 1, so even though \( a \approx b \),

\[
f(a) \neq f(b)
\]

This means that \( f \) is discontinuous at 0.
1.6 Continuity Problems

1. Show that the function $u(x)$ which is 0 when $x \leq 0$ and 1 when $x > 0$ is continuous at $x = 2$ but discontinuous at $x = 0$.

2. Use the angle sum formula

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

and the angle difference formula

$$\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$$

to show that $\cos(\theta)$ is continuous.

3. Show that $h(x) = |x|$ is continuous at $x = 0$.

4. Show that $f(x) = e^x$ is continuous. You may need to use the fact (which you should already have remembered!!) that $e^{-x} = 1/e^x$. 

1.7 Differentiability

Remember that the derivative of a function $f$ measures the slope of the tangent line to the graph of $f$. How do we know that $f$ always has a tangent line? One disaster would be if $f$ had a discontinuity – it would be impossible to find a tangent line at this point. So we know that if $f$ is to be differentiable (meaning “has a derivative”) then at least $f$ must be continuous.

But there is a more subtle sort of disaster that could happen. What if the graph of $f$ had a corner, so that no matter how far we zoomed in it never starts to look like a straight line? At a point where $f$ has a corner, even if $f$ is continuous we can’t define a derivative.

What does it mean to say that the slope of the tangent line is $f'(x)$? In our symbolic language, that is just the formula

$$\frac{f(x + dx) - f(x)}{dx} = f'(x)$$

which defines the derivative. We can rearrange this to get another familiar equation

$$f(x + dx) = f(x) + f'(x) \cdot dx$$

Saying that $f$ “looks linear near $x$” means that for any $y$ which is close to $x$ ($y \approx x$) then the slope

$$\frac{f(y) - f(x)}{y - x}$$

is always the same (draw a picture to see why!). In particular, this slope must be the derivative:

$$\frac{f(y) - f(x)}{y - x} = f'(x) \quad \text{whenever } x \approx y$$

Another way of restating this is by saying that

$$\frac{f(x + n \cdot dx) - f(x)}{n \cdot dx}$$

computes the derivative of $f$ no matter what $n$ is, assuming $f$ is differentiable at $x$.

Let us rearrange this equation a little bit:

$$f(x + n \cdot dx) = f(x) + f'(x) \cdot n \cdot dx$$

This motivates the definition of the derivative: we will say that “$f$ is differentiable at $x$” if
1. \( f \) is continuous at \( x \).

2. \( f \) is infinitessimally linear at \( x \). In symbols, this means that for any number \( n \),

\[
 f(x + n \, dx) = f(x) + f'(x) \cdot n \, dx
\]

Here are two examples to show you how to apply this definition:

**Example:** Let us try to show that \( f(x) = x^2 \) is differentiable. We have already showed that it is continuous, so step 1 is finished. We have also already computed that \( f'(x) = 2x \). Now, we must compute \( f(x + n \, dx) \):

\[
 f(x + n \, dx) = (x + n \, dx)^2 \\
 = x^2 + x \cdot n \, dx + n \, dx \cdot x + (n \, dx)^2 \\
 = x^2 + 2xn \, dx \\
 = f(x) + f'(x) \cdot n \, dx
\]

**Example:** Now let us show that \( h(x) = |x| \) is not differentiable when \( x = 0 \). Again, we already showed that \( |x| \) is continuous at \( x = 0 \), so step 1 is finished. Now let us plug in \( 0 + n \, dx \):

\[
 h(0 + n \, dx) = |0 + n \, dx| = \begin{cases} 
 n \, dx & \text{if } n > 0 \\
 0 & \text{if } n = 0 \\
 -n \, dx & \text{if } n < 0 
\end{cases}
\]

If we looked at \( h(0 + n \, dx) \) with \( n \) positive, we would be led to believe that the derivative was +1. But if we looked at \( h(0 + n \, dx) \) with \( n \) negative, we would be led to believe that the derivative was −1! Therefore, the derivative must not exist at \( x = 0 \).
1.8 Differentiability Problems

1. Show that \( f(x) = x^2 + 3x + 1 \) is differentiable for every \( x \).

2. Is the function defined by
\[
h(x) = \begin{cases} 
-1 & \text{if } x \leq 0 \\
1 & \text{if } x > 0 
\end{cases}
\]
differentiable at \( x = 0 \)? If so, what is its derivative? If not, why not?

3. Show that \( \cos(x) \) and \( \sin(x) \) are differentiable for every \( x \). You will need the angle-sum formulas for both \( \cos \) and \( \sin \), the angle-difference formula for \( \cos \) and the angle-difference formula
\[
\sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \sin(\beta) \cos(\alpha)
\]
for \( \sin \).

4. Can there be a function \( j(x) \) which is continuous at every \( x \), but not differentiable at any \( x \)? If so, what would it look like? If not, why not?