

# CALCULUS ON CATEGORIES

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ABSTRACT. In some geometrically interesting cases, a structure on a manifold  $M$  can be lifted to a related structure on the groupoid  $\mathbf{M}^{(1)}$  of paths on  $M$ . The well-known relation between connections on  $M$  and parallel transport operators provides a motivating example. We examine the problem of lifting geometric constructions such as principal bundles with connection to path space. Additionally, we undertake a sort of inverse problem: determining the geometric structure on  $M$  which corresponds to a connection on  $\mathbf{M}^{(1)}$ . This allows us to infer the existence of a higher categorical structure when certain geometric entities appear on  $M$ . In the process, we develop the basics of calculus on smooth categories, focusing on the central role played by functorial differential forms.

## 1. The Smooth Path Groupoid of $M$

It frequently happens in differential geometry that a certain structure is best thought about as living on the space of paths in  $M$ . Nevertheless, calculations with such a structure are usually done with a more complex (but equivalent) structure on  $M$  itself. For example, a geometer might *define* a connection as a certain type of first-order differential operator, or a kind of  $G$ -equivariant  $n$ -plane distribution, or any of a myriad of other “first order” definitions. Yet when it comes to reasoning geometrically about a connection, we often think of a “zeroth-order” object: the associated parallel transport operator, a particularly nice smooth function on the space of paths in  $M$ . Another fine example is given in [Brylinski], where a volume form on a 3-manifold  $M$  induces a sort of symplectic structure on the space of knots in  $M$ . These lifts to path space are geometrically pleasing, but the functional analysis involved in actually computing on a space like the smooth paths in  $M$  is frequently off-putting.

The philosophy of this paper is that many interesting geometric structures are more easily understood on the path space  $\mathbf{M}^{(1)}$  of  $M$  rather than on  $M$  itself. We would then like to know when a certain structure on  $M$  corresponds to something nice on  $\mathbf{M}^{(1)}$ . Conversely, we should know what common geometric structures on  $\mathbf{M}^{(1)}$  look like on  $M$ . It would also be handy to have computational tools that let us compute directly with the geometric structures on  $\mathbf{M}^{(1)}$  when they appear.

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We will focus on two related problems. The first is fundamental: what are the right generalizations of “differential forms” and “exterior derivative” for computing on  $\mathbf{M}^{(1)}$ ? The second is a recurring problem in differential geometry and mathematical physics: how does the geometry governing trivializations, connections, and curvature lift to path space? Along with this is a sort of inverse question: what geometric data on  $M$  can be associated to a connection on  $\mathbf{M}^{(1)}$ ?

Throughout this paper, we will let  $M$  denote a finite-dimensional smooth manifold. The source and target projections in any category will be denoted  $s$  and  $t$ , respectively.  $G$  will always denote a finite-dimensional Lie group, and  $\text{Hom}_G(X, Y)$  the  $G$ -equivariant maps from  $X$  to  $Y$ .

Because this paper is concerned with local problems (the “calculus” as opposed to the “geometry”), we assume that the manifold  $M$  is contractible. The analysis of these problems when  $M$  has nontrivial topology is the subject of the forthcoming paper *Geometry on Categories*.

Before we begin, let us dispatch with the question of just *which* path groupoid to use. In order to leverage the functional analysis of [Gross], we will let  $\mathbf{M}^{(1)}$  denote the category of piecewise smooth paths on  $M$ . Although most of our computations will take place on  $\mathbf{M}^{(1)}$ , we are only interested in parametrization-independent constructions. We use  $\mathbf{M}^{(1)}$  to mean the category of paths modulo reparametrization. This is not quite a groupoid since we do not factor out by thin homotopy, though it will turn out that all our constructions are thin-homotopy invariant. For most purposes the path groupoid could be replaced with a loop space of  $M$  or the category of smooth holonomic paths in the infinite jet bundle  $J^\infty M \rightarrow M$  with only minor changes to the theorems. In fact, part of the appeal in the categorical approach to path space geometry is that analytic nuances appear to be somewhat irrelevant, replaced by algebraic constraints.

## 2. Functorial de Rham Theory

In this section, we analyze the lift of the de Rham sequence to path space. Throughout this section,  $\Omega_M^k$  will denote the sheaf of  $\mathbb{C}$ -valued differential  $k$ -forms on  $M$ . For the purpose of computation it is useful to work on the category of parameterized paths  $\mathbf{M}^{(1)}$ . Pullback by the functor  $\mathbf{M}^{(1)} \xrightarrow{\mathcal{F}} \mathbf{M}^{(1)}$  which forgets parametrization is a map of sheaves  $\Omega_{\mathbf{M}^{(1)}} \xrightarrow{\mathcal{F}^*} \Omega_{\mathbf{M}^{(1)}}$ . The parametrization-independent forms  $\omega \in \Omega_{\mathbf{M}^{(1)}}$  are exactly the image of  $\mathcal{F}^*$ , and are therefore characterized by

$$(i_X \omega)_\gamma = 0$$

for all vectorfields  $X \in \ker d\mathcal{F}|_\gamma$ . Geometrically, such vectorfields are tangent to the curve  $\gamma$  and vanish on the boundary. A flow in the direction of  $X$  does not change the image of  $\gamma$ , so they project to the zero vector on  $T_\gamma \mathbf{M}^{(1)}$ .

The simplest way to obtain a  $k$ -form on path space is by integrating away a degree of freedom from a  $(k + 1)$ -form on  $M$ .

**2.1. DEFINITION.** *Let  $\omega \in \Omega_M^{k+1}$  and  $\gamma : [0, 1] \rightarrow M$ . Then the transgression of  $\omega$  is the  $k$ -form*

$$(\tau\omega)_\gamma(v_1, \dots, v_k) = \int_0^1 \omega_{\gamma(t)}\left(\frac{d\gamma}{dt}(t), v_1(t), \dots, v_k(t)\right) dt$$

where  $v_i \in T_\gamma \mathbf{M}^{(1)}$ .

When working on  $\mathbf{M}^{(1)}$ , we will only be interested in functorial constructions. In particular, we wish to single out a subset of differential forms on path space which behave nicely with respect to the underlying categorical structure.

**2.2. DEFINITION.** *A  $k$ -form  $\omega \in \Omega_{\mathbf{M}^{(1)}}^k$  is called functorial when*

$$\omega_{\beta \circ \alpha}(X) = \omega_\alpha(X|_\alpha) + \omega_\beta(X|_\beta)$$

and

$$\omega_{\alpha^{-1}} = -\omega_\alpha$$

for all composable curves  $\alpha, \beta$  and all  $k$ -plane fields  $X$  on  $\beta \circ \alpha$ .

The name is justified by the fact that the exterior derivative of a functor is a functorial 1-form. Since we will only be interested in functorial constructions for the remainder of this paper, we adopt the convention that  $\Omega_{\mathbf{C}}^k$  is the sheaf of functorial  $k$ -forms whenever  $\mathbf{C}$  is a smooth category.

We already have a large source of functorial differential forms:

**2.3. LEMMA.** *The transgression of a  $k$ -form is functorial, so  $\tau$  defines a map*

$$\Omega_M^k \xrightarrow{\tau} \Omega_{\mathbf{M}^{(1)}}^{k+1}$$

**PROOF.** Immediate from the definition of the integral. ■

Functorial  $k$ -forms have a local nature to them, since the value at  $\beta \circ \alpha$  can be directly computed from the values at  $\beta$  and  $\alpha$  individually and independently. This means that any functorial  $k$ -form is determined by an infinitesimal neighborhood of the map which takes  $x \in M$  to the constant path  $\bar{x}$ . In particular, it should mean that functorial  $k$ -forms can be described as sections of some bundle over  $M$  itself.

Heuristically, the idea is as follows. By making an infinitely fine subdivision of each path, we can describe a functorial  $k$ -form on  $\mathbf{M}^{(1)}$  by a standard  $k$ -form on  $M$  with an extra input for an infinitesimal path. But tangent vectors are the germs of paths, so we should end up with something like a  $(k + 1)$ -form on  $M$  describing “evaluation on infinitesimal paths” for our functorial  $k$ -form on  $\mathbf{M}^{(1)}$ . Since this  $(k + 1)$ -form carries all of the information about the original  $k$ -form, the map should be both left and right invertible. The following theorems justify this intuition:

2.4. THEOREM. Let  $\omega \in \Omega_{\mathbf{M}^{(1)}}^k(\mathfrak{g})$  be a functorial and parametrization-independent  $k$ -form. Then there is a  $(k+1)$ -form on  $M$  defined by

$$(\varepsilon\omega)_x(v_0, \dots, v_k) = \lim_{t \rightarrow 0^+} \frac{1}{t} \cdot \omega_{\gamma|_{[-1,t]}}(\tilde{v}_1, \dots, \tilde{v}_k)$$

where  $\gamma$  is any smooth path parametrized by  $[-1, 1]$  with  $\gamma|_{[-1,0]}$  thin-homotopic to  $\bar{x}$ ,  $\dot{\gamma}_0 = v_0$ , and where  $\tilde{v}_i$  is any extension of  $v_i$  to a vectorfield along  $\gamma$  with

$$\omega_{\gamma|_{[-1,0]}}(\tilde{v}_1, \dots, \tilde{v}_k) = 0$$

PROOF. The functoriality equation for  $\omega$  implies that  $\omega_{\gamma|_{[-1,t]}}$  is on the order of  $t$ , so the limit is at least finite. This also means that only the first-order germ of  $\gamma$  contributes to the limit, so the choice of path  $\gamma$  extending  $v_0$  is irrelevant. Now suppose that  $v'_1, \dots, v'_k$  is an alternate extension of  $\tilde{v}_1, \dots, \tilde{v}_k$ , and expand the difference in powers of  $t$ :

$$v'_i - \tilde{v}_i = w_i \cdot t + o(t^2)$$

There are no terms of order zero since  $v'_i$  also extends  $v_i$ . We then have

$$\omega_{\gamma|_{[-1,t]}}(v'_1, \dots, v'_k) = \omega_{\gamma|_{[-1,t]}}(\tilde{v}_1, \dots, \tilde{v}_k) + t \cdot \sum_i \omega_{\gamma|_{[-1,t]}}(\tilde{v}_1, \dots, w_i, \dots, \tilde{v}_k) + o(t^2)$$

However, since  $\omega_{\gamma|_{[-1,t]}}$  is itself of order  $t$ , the only term which contributes to the limit is the first.

We have therefore shown that the limit is well-defined, but *a priori* we only end up with a section of  $TM^* \otimes \Omega_M^k$ . To get antisymmetry, it suffices to show that  $(\varepsilon\omega)_x(v_0, v_0, v_1, \dots, v_{k-1}) = 0$ . After choosing  $\gamma$  extending  $v_0$ , extend  $v_0$  to a vectorfield  $T$  tangent to  $\gamma$  and vanishing on the endpoints<sup>1</sup>. The limit defining  $\varepsilon\omega$  is

$$\lim_{t \rightarrow 0^+} \omega_{\gamma|_{[-1,t]}}(T, \dots)$$

but since  $\omega$  is parametrization-independent,  $i_T\omega = 0$ . Thus,  $\varepsilon\omega$  is totally antisymmetric.

PROOF.

2.5. COROLLARY. The map  $\varepsilon$  descends to  $\mathbf{M}^{(1)}$ .

2.6. COROLLARY.  $\varepsilon$  and  $\tau$  are two-sided inverses.

<sup>1</sup>This is possible only because we chose  $\gamma$  with  $\gamma|_{[-1,0]}$  thin-homotopic to a trivial path. This allows us to extend  $v_0$  without otherwise affecting  $\omega$ , since for the entire proof, the value of  $\omega$  on  $[-1, t]$  is equal to its value on  $[0, t]$ . The author apologizes for the tastelessness of this detail.

PROOF. Throughout, we let  $x$  and  $\gamma$  be as in the above theorem, and  $v = v_0$ . Let  $\omega \in \Omega_M^k$  be given. Then

$$(i_v(\varepsilon\tau\omega))_x = \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t (i_{\frac{d\gamma}{du}}\omega)_{\gamma_u} du = (i_v\omega)_x$$

so  $\varepsilon\tau = \text{id}$ .

Now let  $\eta \in \Omega_{M^{(1)}}^k$ . By functorality and the definition of the Riemann integral,

$$\eta_\gamma = \int_0^1 \left( i_{\frac{d\gamma}{du}}(\varepsilon\eta) \right)_{\gamma_u} du = (\tau\varepsilon\eta)_\gamma$$

PROOF. For calculus on categories, the exterior derivative is not sufficient. Instead, we would like a differential operator which respects the categorical structure. In particular, the derivative should have a component “inside  $\text{Hom}(x, y)$ ” and also a component describing the “motion of  $x$  and  $y$ ”. In the case of  $M^{(1)}$ ,

2.7. DEFINITION. *The functorial exterior derivative is defined by*

$$d = \varepsilon|_{\partial} - d$$

where  $|_{\partial}$  means the evaluation at the endpoints. Explicitly, if  $X = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k T_\gamma M^{(1)}$  then

$$(d\omega)_\gamma(X) = (\varepsilon\omega)_{t\gamma}(dt(X)) - (\varepsilon\omega)_{s\gamma}(ds(X)) - d\omega_\gamma(X)$$

where  $d$  is the standard exterior derivative.

2.8. THEOREM. *The diagram*

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & \underline{\mathbb{C}}_M & \xrightarrow{\iota} & \Omega_M^0 & \xrightarrow{d} & \Omega_M^1 & \xrightarrow{d} & \Omega_M^2 & \xrightarrow{d} & \Omega_M^3 & \xrightarrow{d} & \cdots \\ & & \varepsilon \uparrow \downarrow \tau & & \\ 1 & \longrightarrow & \underline{\mathbb{C}}_{M^{(1)}} & \xrightarrow{\iota} & \underline{\mathbb{C}}_{M^{(1)}} & \xrightarrow{\iota} & \Omega_{M^{(1)}}^0 & \xrightarrow{d} & \Omega_{M^{(1)}}^1 & \xrightarrow{d} & \Omega_{M^{(1)}}^2 & \xrightarrow{d} & \cdots \end{array}$$

commutes, where  $\underline{\mathbb{C}}_M$  is the sheaf of locally constant functions on  $M$ ,  $\underline{\mathbb{C}}_{M^{(1)}}$  the sheaf of locally constant functions on  $M^{(1)}$  and  $\underline{\mathbb{C}}_{M^{(1)}}$  the sheaf of smooth functions which are locally constant on each  $\text{Hom}$ -set of  $M^{(1)}$ .

PROOF. Let  $\omega \in \Omega_M^k$  be given and fix a  $[0, 1]$ -parametrized path  $\gamma \in \mathbf{M}^{(1)}$  to focus on. Without loss of generality, we may choose coordinates  $x_0, x_1, \dots$  such that  $\partial/\partial x^0 = d\gamma/dt$ . In these coordinates,  $\omega$  takes the form

$$\omega = \sum_{|I|=k-1, 0 \notin I} \omega_{0,I} dx^0 \wedge dx^I + \sum_{|J|=k, 0 \notin J} \omega_J dx^J$$

where  $I$  and  $J$  are multi-indices. Let us first compute  $\tau d\omega$ . The derivative is

$$d\omega = \sum_{\substack{|I|=k-1, 0 \notin I \\ i \geq 1}} -\frac{\partial \omega_{0,I}}{\partial x^i} dx^0 \wedge dx^i \wedge dx^I + \sum_{|J|=k, 0 \notin J} \frac{\partial \omega_J}{\partial x^0} dx^0 \wedge dx^J + \dots$$

where the unwritten terms do not involve  $dx^0$  and will therefore vanish under integration. Since  $d\gamma = \frac{\partial}{\partial x^0} dt$ ,

$$\int_{\gamma} \frac{\partial f}{\partial x^0} dx^0 = f(t\gamma) - f(s\gamma)$$

so we have

$$\tau d\omega = \sum_{\substack{|I|=k-1, 0 \notin I \\ i \geq 1}} \left( -\int_0^1 \frac{\partial \omega_{0,I}}{\partial x^i} dt \right) dx^i \wedge dx^I + \sum_{|J|=k, 0 \notin J} \omega_J|_{\partial\gamma} dx^J$$

On the other hand,

$$\tau\omega = \sum_{|I|=k-1, 0 \notin I} \left( \int_0^1 \omega_{0,I} dt \right) dx^I$$

Using the fact that  $\varepsilon$  inverts  $\tau$ , we have

$$d\tau\omega = -d\tau\omega + \omega|_{\partial\gamma} = \sum_{\substack{|I|=k-1, 0 \notin I \\ i \geq 0}} \left( -\int_0^1 \frac{\partial \omega_{0,I}}{\partial x^i} dt \right) dx^i \wedge dx^I + \sum_{|J|=k} \omega_J|_{\partial\gamma} dx^J$$

The  $i = 0$  terms in the first sum are cancelled by the  $0 \in J$  terms from the second sum, so  $d\tau\omega = \tau d\omega$ . ■

2.9. COROLLARY.  $d^2 = 0$ .

### 3. The Problem in the Nonabelian Case

Many interesting geometric problem can be formulated as statements about connections on principal  $G$ -bundles over  $M$ , where  $G$  is a Lie group. We would like to describe how these problems appear when lifted to path space, and also find the “infinitesimal” version of a principal  $G$ -bundle with connection on path space. In the language of the previous section, we would like to describe nonabelian transgression and infinitesimal evaluation maps.

There are several problems which prevent this from being straightforward. In the case of a nonabelian Lie group  $G$ , there is no nice analog of the de Rham sequence in all degrees. We only have the sheaf morphisms

$$1 \longrightarrow \underline{G}_M \xrightarrow{\iota} C_M^\infty(G) \xrightarrow{* \theta} \Omega_M^1(\mathfrak{g}) \xrightarrow{\text{curv}} \Omega_M^2(\mathfrak{g})$$

where  $f^*\theta = f^{-1}df$  is the pullback of the Maurer-Cartan form, and  $\text{curv } \omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$  computes the curvature of  $\omega$ . These maps satisfy  $(\iota c)^*\theta = 0$  and  $\text{curv } (f^*\theta) = 0$ . The sequence is almost exact, in the sense that if the image of a section  $\xi$  is trivial then  $\xi$  is in the image of the previous map. Even more, by the fundamental theorem of calculus if  $f^*\theta = g^*\theta$  then  $f = (\iota c) \cdot g$ . Unfortunately, there are problems in the next degree: even if  $\text{curv } \omega = \text{curv } \eta$  it can happen that  $\omega$  and  $\eta$  are not related by a gauge transformation from  $C_M^\infty(G)$  (which is the nonabelian version of exactness). The existence of these nonequivalent connections with identical curvatures was studied in [Wu] and [Gross] as the *field copy problem*. It arises naturally when studying the relation between field strengths and gauge potentials in Yang-Mills theory. In [Gross], the non-exactness of the sequence at this point is interpreted to mean that the curvatures of a nonabelian Yang-Mills field do not yield a complete set of observables. This is in stark contrast to the abelian case of electromagnetism, where measuring the field strength (curvature) gives us all observable information about the electromagnetic potential. To handle the nonabelian case, [Gross] proposes an alternate set of observables, which in the language of this paper form a functorial, nonabelian connection on a principal  $G$ -bundle over path space. A related theme is found in [Polyakov] when the author describes a string-theoretic physics on  $M$  by a gauge-theoretic physics on  $M^{(1)}$ .

Of course we may still want to know if a certain 2-form  $\omega \in \Omega_M^2(\mathfrak{g})$  is the curvature of *any* connection, even disregarding the aforementioned issues of uniqueness. But even this simpler question has serious problems. In the abelian case, a 2-form  $\omega$  is a curvature if and only if  $d\omega = 0$ . The analogous nonabelian condition is the Bianchi identity  $d^A(\text{curv } A) = 0$ , where

$$d^A\omega = d\omega + [A \wedge \omega]$$

But this equation cannot even be formulated without already knowing a connection  $A$  with  $\text{curv } A = \omega$ . There is no other obvious necessary condition to determine if a 2-form is a curvature, so we appear to be stuck.

To summarize, there are two problems with the nonabelian de Rham sequence at  $\Omega_M^1(\mathfrak{g}) \xrightarrow{\text{curv}} \Omega_M^2(\mathfrak{g})$ . The first is the lack of a map  $\Omega_M^2(\mathfrak{g}) \xrightarrow{d_\eta} \Omega_M^3(\mathfrak{g})$  with  $d_\eta^{-1}(0) = \text{im curv}$ . The second problem is that the equality  $\text{curv } \omega = \text{curv } \eta$  does not imply the gauge equivalence

$$\eta = \text{Ad}(f^{-1})(\omega) + f^*\theta$$

with  $f \in C_M^\infty(G)$ . Since gauge equivalence is the nonabelian version of cohomologous, any notion of exactness is ruined.

Undaunted by these problems, let us look for a “morally correct” way around them. If the  $\tau$  and  $\varepsilon$  maps from the previous section could be given nonabelian generalizations, then the composition

$$\begin{array}{ccc} \Omega_M^2(\mathfrak{g}) & & \Omega_M^3(\mathfrak{g}) \\ \downarrow \tau & & \uparrow \varepsilon \\ \Omega_{M^{(1)}}^1(\mathfrak{g}) & \xrightarrow{\text{curv}} & \Omega_{M^{(1)}}^2(\mathfrak{g}) \end{array}$$

would provide a function which vanishes precisely on those 2-forms which are curvatures. Still better, recall that  $\text{curv } \eta = 0$  implies  $\eta = F^*\theta$  for a unique  $F$  up to gauge equivalence. Therefore when  $\varepsilon(\text{curv } (\tau\omega)) = 0$ ,  $\tau\omega = F^*\theta$  and  $\varepsilon F$  is a connection on  $M$  with curvature  $\omega$ . For this last computation, we utilized the commutativity of the (unfortunately nonexistent) diagram

$$\begin{array}{ccc} \Omega_M^1(\mathfrak{g}) & \xrightarrow{\text{curv}} & \Omega_M^2(\mathfrak{g}) \\ \varepsilon \uparrow & & \downarrow \tau \\ C_{M^{(1)}}^\infty(G) & \xrightarrow{*\theta} & \Omega_{M^{(1)}}^1(\mathfrak{g}) \end{array}$$

Of course, the problems at  $\Omega_M^1(\mathfrak{g}) \xrightarrow{\text{curv}} \Omega_M^2(\mathfrak{g})$  prevent us from implementing the solution just described. Nevertheless, in the next two sections we will approach the problem by describing the functorial nonabelian de Rham sequence on  $M^{(1)}$  and analyzing its infinitesimal behavior. This will eventually lead us to a solution of a related problem, with  $G$  replaced by a certain 2-group.

## 4. Functorial $G$ -Bundles

In this section, we allow  $M$  to have nontrivial topology.

The classical nonabelian analog of transgression is the equivalence between  $G$ -connections and  $G$ -valued parallel transport operators. Parallel transport describes a  $G$ -equivariant motion in a bundle from the fiber over one point to the fiber over another point, determined by the connection and the path chosen to between the points. Locally, if the connection form is  $\omega \in \Omega_M^1(\mathfrak{g})$  then the associated parallel transport operator is

$$P^\omega(\gamma) = \exp \int_\gamma \omega$$

where  $\exp \int$  is the path-ordered product. If we assume  $\gamma$  is  $[0, 1]$  parametrized, the explicit definition is

$$\exp \int_\gamma \omega = \lim_{N \rightarrow \infty} \prod_{k=1}^N \exp \left( \frac{1}{N} \cdot \omega_{\gamma(\frac{k}{N})} \left( \frac{d\gamma}{dt} \left( \frac{k}{N} \right) \right) \right)$$

The exp on the right-hand side of the equation is the exponential map  $\mathfrak{g} \xrightarrow{\exp} G$ . Since parallel transport operators are related to connections by this exponential transgression map, they are a natural starting point for understanding the nonabelian case. To this end, let us spend some time understanding the bundles on which parallel transport operators live in an invariant way.

To each bundle  $G \rightarrow E \xrightarrow{\pi} M$  there is an associated trivial bundle  $G \rightarrow E_{rel} \xrightarrow{\pi_{rel}} \mathbf{M}^{(1)}$  with fiber

$$\pi_{rel}^{-1}(\gamma) = \text{Hom}_G(\pi^{-1}(s\gamma), \pi^{-1}(t\gamma))$$

The bundle  $E_{rel}$  is *functorial* in the following sense. Let  $\alpha, \beta \in \mathbf{M}^{(1)}$ , be a pair of composable paths, and take  $x \in \pi_{rel}^{-1}(\alpha)$ ,  $y \in \pi_{rel}^{-1}(\beta)$ . Then there is a composite element  $y \circ x \in \pi_{rel}^{-1}(\beta \circ \alpha)$  due to the natural map

$$\text{Hom}_G(y, z) \times \text{Hom}_G(x, y) \xrightarrow{\circ} \text{Hom}_G(x, z)$$

Any connection  $\nabla$  on  $E$  induces a parallel transport operator

$$P^\nabla \in \Gamma(E_{rel} \rightarrow \mathbf{M}^{(1)})$$

Since  $P^\nabla(\beta \circ \alpha) = P^\nabla(\beta) \circ P^\nabla(\alpha)$ , the section of  $E_{rel}$  corresponding to parallel transport respects the functorial structure of the bundle.

To summarize, each  $G$ -bundle  $E$  on  $M$  induces a *trivial* and *functorial* bundle  $E_{rel}$  over  $\mathbf{M}^{(1)}$ . Each connection  $\nabla$  on  $E$  induces a functorial section  $P^\nabla$  of  $E_{rel}$ .

The construction is entirely reversible:

4.1. THEOREM. *If  $G \rightarrow \hat{E} \xrightarrow{\hat{\pi}} \mathbf{M}^{(1)}$  is a trivial functorial bundle then there is an associated bundle  $G \rightarrow E \xrightarrow{\pi} M$  such that  $E_{rel} \cong \hat{E}$ . Furthermore, to every functorial trivialization  $P$  of  $\hat{E}$  there is a unique connection  $\nabla$  on  $E$  with  $P^\nabla = P$ .*

PROOF. Let  $\hat{E}$  be as above, and choose a Čech cover  $\{U_i\}$  on  $M$  along with marked points  $x_i \in U_i$ . On each  $U_i$ , pick a local trivialization by selecting a function  $\psi_i$  with  $\psi_i(x) \in \text{Hom}_G(x_i, x)$ . The local trivializations are related by the cochain<sup>2</sup>

$$\psi_j^{-1} \circ \psi_i \stackrel{\text{def}}{=} \varphi_{ij} : U_i \cap U_j \longrightarrow \text{Hom}_G(x_i, x_j)$$

$\varphi_{ij}$  satisfies the cocycle condition

$$\text{id} = (\partial\varphi)_{ijk} = \varphi_{ki} \circ \varphi_{jk} \circ \varphi_{ij} \in \text{Hom}_G(x_i, x_i)$$

and therefore defines a principal  $G$ -bundle on  $M$ .

Now let  $P$  be a functorial section of  $\hat{E}$ . We will construct a connection on  $E$  by describing a path-lifting procedure derived from  $P$ . First, note that for the constant paths  $\bar{x}$ ,

$$P(\bar{x}) = \text{id} \in \text{Hom}_G(x, x)$$

For any path  $\gamma : [0, 1] \longrightarrow M$ , let  $\gamma^t(s) = \gamma(ts)$ . For each path  $\gamma$  on  $M$  we define a lift into  $E$  as follows. Pick  $g_0 \in \pi^{-1}(\gamma(0))$  and define

$$\gamma_P(t) = P(\gamma^t)(g_0)$$

$\gamma_P(0) = g_0$  and  $\gamma_P(t) \in \pi^{-1}(\gamma(t))$ . Furthermore, this lift is constructed to be equivariant with respect to the choice of  $g_0$ . In other words,  $P$  is the parallel transport of a connection  $\nabla$  on  $E$ . ■

4.2. COROLLARY. *To each functor  $\mathbf{M}^{(1)} \xrightarrow{F} G$  there is a unique principal  $G$ -bundle  $E$  with connection  $\nabla$  on  $M$  such that  $F = P^\nabla$ .*

PROOF. Since  $F$  is a functor, it gives a functorial trivialization of the trivial bundle  $\hat{E} = \mathbf{M}^{(1)} \times G$ . ■

## 5. Nonabelian Infinitesimal Functors

We now would like to generalize the notion of “functorial  $k$ -form” to the nonabelian case, at least for small values of  $k$ . For  $k = 1$ , a functorial 1-form should be algebraically the same as the derivative of a functor. Given a smooth functor  $\mathbf{M}^{(1)} \xrightarrow{f} G$ , the Maurer-Cartan form  $f^*\theta$  satisfies the *infinitesimal functorality* condition

$$f^*\theta_{\beta \circ \alpha} = f^*\theta_\alpha + \text{Ad}(f_\alpha^{-1})(f^*\theta_\beta)$$

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<sup>2</sup>Readers who are surprised by the appearance of the  $G$ -torsor  $\text{Hom}_G(x_i, x_j)$  rather than  $G$  itself may note that  $\varphi_{ij}$  can be turned into a  $G$ -cochain in the usual sense by choosing an explicit isomorphism of the fiber over each  $x_i$  with  $G$ .

5.1. DEFINITION. A functorial 1-form  $\omega$  on  $\mathbf{M}^{(1)}$  is a 1-form  $\omega \in \Omega_{\mathbf{M}^{(1)}}^1(\mathfrak{g})$  and a functor  $\mathbf{M}^{(1)} \xrightarrow{f} G$  which satisfy

$$\omega_{\beta \circ \alpha} = \omega_\alpha + \text{Ad}(f_\alpha^{-1})(\omega_\beta)$$

More generally, if  $\mathbf{G}$  is a 2-group<sup>3</sup> with  $G = \text{Hom}(1, -)$ ,  $H = \text{ob } \mathbf{G}$ , and  $\Phi$  the action of  $H$  on  $G$  then a  $\mathbf{G}$ -valued 1-form on  $\mathbf{M}^{(1)}$  is given by a 1-form  $\omega \in \Omega_{\mathbf{M}^{(1)}}^1(\mathfrak{g})$  and a functor  $f : M \rightarrow H$  which satisfy

$$\omega_{\beta \circ \alpha} = \omega_\alpha + \Phi(f_\alpha^{-1})(\omega_\beta)$$

$G$ -valued functorial 1-forms are therefore the same as  $\text{Ad}_G$ -valued 1-forms, where  $\text{Ad}_G$  is the sub-2-group of inner automorphisms in  $\text{Aut}_G$ . This perspective becomes important when we examine the image of functorial forms under  $\varepsilon$ .

There is no difficulty extending the map  $\varepsilon$  to the nonabelian case.

5.2. THEOREM. For any Lie group  $G$ , the following diagram of sheaves

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \underline{G}_M & \xrightarrow{\iota} & C_M^\infty(G) & \xrightarrow{*\theta} & \Omega_M^1(\mathfrak{g}) & \xrightarrow{\text{curv}} & \Omega_M^2(\mathfrak{g}) & & \Omega_M^3(\mathfrak{g}) \\ & & \varepsilon \uparrow \downarrow \tau & & \varepsilon \uparrow \downarrow \tau & & \varepsilon \uparrow \downarrow \tau & & \varepsilon \uparrow & & \varepsilon \uparrow \\ 1 & \longrightarrow & \underline{G}_{\mathbf{M}^{(1)}} & \xrightarrow{\iota} & C_{\mathbf{M}^{(1)}}^\infty(G) & \xrightarrow{*\theta} & \Omega_{\mathbf{M}^{(1)}}^1(\mathfrak{g}) & \xrightarrow{\text{curv}} & \Omega_{\mathbf{M}^{(1)}}^2(\mathfrak{g}) & & \end{array}$$

commutes, and following any two horizontal arrows results in the trivial map.

The problem arises when we try to define a transgression map  $\Omega_M^2(\mathfrak{g}) \xrightarrow{\tau} \Omega_{\mathbf{M}^{(1)}}^1(\mathfrak{g})$ . Because of the field copy problem, a nonabelian curvature does not carry enough information to reconstruct its connection. Yet if we could lift the curvature to path space, a connection could always be found. Therefore, it is impossible to find a good  $\Omega_M^2(\mathfrak{g}) \xrightarrow{\tau} \Omega_{\mathbf{M}^{(1)}}^1(\mathfrak{g})$ . To solve this problem, let us analyze the infinitesimal evaluation map  $\Omega_{\mathbf{M}^{(1)}}^1(\mathfrak{g}) \xrightarrow{\varepsilon} \Omega_M^2(\mathfrak{g})$  more carefully.

In the following, we let  $\omega = (\omega, \psi)$  be a functorial 1-form on  $\mathbf{M}^{(1)}$ . When we apply the infinitesimal evaluation map  $\varepsilon$  to  $\omega$  and obtain a 2-form on  $M$ , it is clear that some information will be lost:  $\omega' = (\omega, \psi)$  and  $\omega' = (\omega, \psi')$  will evaluate to the same 2-form even if they do not represent equivalent connections over  $\mathbf{M}^{(1)}$ . If we want to be able to reconstruct  $\omega$ , we must also compute  $\varepsilon\psi \in \Omega_M^1(\mathfrak{g}/\mathcal{Z}\mathfrak{g})$ . That is to say, a  $G$ -connection on  $\mathbf{M}^{(1)}$  corresponds on  $M$  to a pair

$$(A, F) \in \Omega_M^1(\mathfrak{g}/\mathcal{Z}\mathfrak{g}) \oplus \Omega_M^2(\mathfrak{g})$$

This is precisely the data defining a 2-connection on an  $\text{Ad}_G$ -2-bundle in the sense of [HGT].

<sup>3</sup>We only concern ourselves with strict 2-groups (crossed modules) in this paper. Extensive background on 2-groups may be found in [HDA5].

5.3. DEFINITION. Let  $\mathbf{G}$  be a 2-group with  $G = \text{Hom}(1, -)$  and  $H = \text{ob } \mathbf{G}$ , and write  $\mathfrak{g}$  for the Lie algebra of  $\mathbf{G}$ . Then a  $\mathfrak{g}$ -valued  $k$ -form on  $M$  is defined by

$$\Omega_M^k(\mathfrak{g}) = \Omega_M^{k-1}(\mathfrak{h}) \oplus \Omega_M^k(\mathfrak{g})$$

5.4. THEOREM. Let the local data for a connective structure on an  $\text{Ad}_G$ -bundle over  $M$  be given as  $A \in \Omega_M^1(\mathfrak{g}/\mathcal{Z}\mathfrak{g})$  and  $F \in \Omega_M^2(\mathfrak{g})$ . Then there is a connection form  $\omega \in \Omega_{\mathbf{M}^{(1)}}^1(\mathfrak{g})$  given by

$$\omega_\gamma(v) = \int_0^1 \text{Ad}\left(\exp \int_{\gamma|_{[0,t]}} A\right)^{-1} \left( F_{\gamma(t)}\left(\frac{d\gamma}{dt}(t), v(t)\right) \right) dt$$

where  $\gamma : [0, 1] \rightarrow M$  and  $v \in T_\gamma \mathbf{M}^{(1)}$ . Furthermore,  $\omega$  is the transgression of  $A$  and  $F$  in the sense that  $\varepsilon\omega = (A, F) \in \Omega_M^2(\text{ad}_G)$ .

PROOF. By definition,  $\omega$  satisfies

$$\omega_{\beta \circ \alpha} = \omega_\alpha + \text{Ad}\left(\left(\exp \int_\alpha A\right)^{-1}\right)(\omega_\beta)$$

so the 1-form part of  $\varepsilon\omega$  is simply  $A$ . As for the 2-form part, just as in the abelian case if  $d\gamma/dt(0) = w$  then

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \int_0^h \text{Ad}\left(\exp \int_{\gamma|_{[0,t]}} A\right)^{-1} \left( F_{\gamma(t)}\left(\frac{d\gamma}{dt}(t), v(t)\right) \right) dt = F_{\gamma(0)}(w, v)$$

■

In other words, whenever an  $\text{Ad}_G$ -2-bundle with 2-connection appears on  $M$ , there is a corresponding  $G$ -bundle with connection on  $\mathbf{M}^{(1)}$ . One could then choose to work with familiar geometric structures on the pathspace of  $M$  or with less familiar (but “more finite”) geometric structures on  $M$  itself, depending on the nature of the problem.

Given the functorial connection  $\omega$ , it is easy to read off geometric information about the related  $\text{Ad}_G$ -valued 2-connection. For example,

5.5. LEMMA. The condition  $\psi^*\theta = dt(\omega)$  is equivalent to the vanishing of the fake curvature of  $\varepsilon\omega$ .

PROOF. Let  $A = \varepsilon\psi$  and  $F = \varepsilon\omega$  be the infinitesimal 1- and 2-form parts of  $\omega$ .  $\varepsilon(\psi^*\theta) = \text{curv } A$ , so the equation  $\psi^*\theta - dt(\omega) = 0$  infinitesimally becomes

$$dA + \frac{1}{2}[A \wedge A] - dt(F) = 0$$

■

5.6. THEOREM. *If the curvature  $d\omega + \frac{1}{2}[\omega \wedge \omega]$  and fake curvature  $\psi^*\theta - dt(\omega)$  vanish then  $\omega = \Psi^*\theta$  for some functor  $\Psi : \mathbf{M}^{(1)} \rightarrow G$  lifting  $\psi$ .*

PROOF. Let  $f : \mathbf{M}^{(1)} \rightarrow G$  be an antiderivative of  $\omega$ , so that  $f^*\theta = \omega$ . The existence of such an  $f$  follows from an extension of Cartan's lemma to path space due to L. Gross. We will abuse notation by picking a lift  $G/\mathcal{Z}G \xrightarrow{\ell} G$  and writing  $\psi$  for  $\ell(\psi)$ . The vanishing of the fake curvature implies that  $f^*\theta = \psi^*\theta \pmod{\mathcal{Z}\mathfrak{g}}$ , so by choosing the constant of integration we may assume that  $f = \lambda \cdot \psi$  for some function  $\lambda : \mathbf{M}^{(1)} \rightarrow \mathcal{Z}G$ . Since  $\lambda$  is central and  $\psi$  is a functor,  $f$  is a functor if and only if  $\lambda$  is. We proceed by differentiating  $f_{\beta\circ\alpha} \cdot f_\alpha^{-1} \cdot f_\beta^{-1}$ , multiplying on the left by  $f_{\beta\circ\alpha}^{-1}$  and on the right by  $f_\beta \cdot f_\alpha$ :

$$\begin{aligned} f_{\beta\circ\alpha}^{-1} \cdot d(f_{\beta\circ\alpha} \cdot f_\alpha^{-1} \cdot f_\beta^{-1}) \cdot f_\beta \cdot f_\alpha &= f^*\theta_{\beta\circ\alpha} - f^*\theta_\alpha - f_\alpha^{-1} \cdot f^*\theta_\beta \cdot f_\alpha \\ &= \omega_{\beta\circ\alpha} - \omega_\alpha - \text{Ad}(f_\alpha^{-1})(\omega_\beta) \\ &= \omega_{\beta\circ\alpha} - \omega_\alpha - \text{Ad}(\psi_\alpha^{-1})(\omega_\beta) = 0 \end{aligned}$$

since  $\omega$  is functorial. It follows that  $f_{\beta\circ\alpha} = cf_\beta \cdot f_\alpha$  for some constant  $c \in G$ . In fact,  $c$  must be central: since  $f_{\bar{x}} = f_{\bar{x}\circ\bar{x}} = cf_{\bar{x}}^2$ ,

$$c = f_{\bar{x}}^{-1} = \lambda_{\bar{x}}^{-1} \cdot \psi_{\bar{x}}^{-1} = \lambda_{\bar{x}}^{-1} \in \mathcal{Z}G$$

Finally, set  $\Psi = cf$ . Then  $\Psi^*\theta = \omega$  and

$$\Psi_{\beta\circ\alpha} = cf_{\beta\circ\alpha} = c^2 f_\beta \cdot f_\alpha = \Psi_\beta \cdot \Psi_\alpha$$

which completes the proof.  $\blacksquare$

The assumption of vanishing fake curvature is somewhat ungainly. In the generic case where the holonomy of  $\omega$  is surjective, vanishing curvature implies vanishing fake curvature. Even in the non-generic cases, when the curvature of  $\omega$  vanishes there is still a functorial antiderivative, but the relation of this antiderivative to  $\psi$  is surprisingly subtle. A detailed account of the situation is to appear in an upcoming paper *Pathspace Geometry and the Field Copy Problem*.

Altogether, this discussion proves

5.7. THEOREM. *For any Lie group  $G$ , the following diagram of sheaves*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \underline{\text{Ad}}_{G/M} & \xrightarrow{\iota} & C_M^\infty(\text{Ad}_G) & \xrightarrow{*\theta} & \Omega_M^1(\text{ad}_G) & \xrightarrow{\text{curv}} & \Omega_M^2(\text{ad}_G) & \xrightarrow{2\text{curv}} & \Omega_M^3(\mathfrak{g}) \\ & & \varepsilon \uparrow \downarrow \tau & & \varepsilon \uparrow \\ 1 & \longrightarrow & \underline{G}_{M^{(1)}} & \xrightarrow{\iota} & \underline{G}_{M^{(1)}} & \xrightarrow{\iota} & C_{M^{(1)}}^\infty(G) & \xrightarrow{*\theta} & \Omega_{M^{(1)}}^1(\mathfrak{g}) & \xrightarrow{\text{curv}} & \Omega_{M^{(1)}}^2(\mathfrak{g}) \end{array}$$

commutes, and following any two horizontal arrows results in the trivial map. Here,

$$2\text{curv}(A, F) = dF + [A \wedge F]$$

It would be impossible to conclude without the obvious conjecture:

5.8. CONJECTURE. *There is a sequence  $A_1 = G$ ,  $A_2 = \text{Ad}_G$ ,  $\dots$  with  $A_k$  an  $n$ -group such that the diagram in the previous theorem extends rightward to  $\Omega_M^{k+1}(\mathfrak{a}_k)$  and downward to  $\mathbf{M}^{(k-1)}$ .*

## References

- [Gross] Leonard Gross, A Poincaré Lemma for Connection Forms, *J. Functional Analysis* *64.1* (1985).
- [Brylinski] Jean-Luc Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhäuser (1992).
- [HDA5] John C. Baez and Aaron D. Lauda, Higher-Dimensional Algebra V: 2-Groups, Theory and Applications of Categories *12*, [math/0307200](#) (2004).
- [HGT] John Baez and Urs Schreiber, Higher Gauge Theory: 2-Connections on 2-Bundles, [hep-th/0412325](#) (2004).
- [Polyakov] A. M. Polyakov, *Gauge Fields and Strings*, CRC Press (1987).
- [Wu] Wu, Tai Tsun, Some remarks about unquantized non-Abelian gauge fields, *Phys. Rev. D* *12.12* (1975), 3843 – 3844.

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