Let $M$ be a Riemann surface and identify $\mathbb{R}^3$ with the imaginary quaternions $\im H$. Then an immersion $M \xrightarrow{f} \mathbb{R}^3$ with unit normal $N$ is conformal if and only if it satisfies the “moving Cauchy-Riemann equations”

$$\star df = N \cdot df$$

so that $f_y = N \cdot f_x$. Given such an $f$, the tangential part of the Laplacian vanishes:

$$(\star d \star d f, df) = (\star dN \wedge d f, d f) = 0$$

since $dN$ is tangential and the quaternion product of two vectors is perpendicular to their span. In fact, we can be more precise: if $\lambda = |f_x|^2 = |f_y|^2$ is the scaling factor of $f$,

$$d \star df = \left( f_{xx} + f_{yy}\right) \perp dx \wedge dy = -2\lambda H dx \wedge dy = -H df \wedge df$$

where $H$ is the mean curvature of $f$. In particular, $f(M) \subset \mathbb{R}^3$ is a minimal surface if and only if $f$ is harmonic, $d \star df = 0$.

From a slightly different perspective, we have that $f$ is minimal iff

$$0 = (\triangle f, N) = -\text{tr}(df, dN) = -f_x N_x - f_y N_y = -f_x (N_x - NN_y)$$

This shows that $f(M)$ is a minimal surface exactly when the map $N : M \rightarrow S^2$ is an antiholomorphic map.

Harmonic maps to $S^2$: Let $(,)$ denote the inner product on the round $S^2$, and define the elastic energy functional

$$E(\varphi) = \frac{1}{2} \int_M (d\varphi \wedge \star d\varphi)$$

on maps $\varphi : M \rightarrow S^2$. We investigate stationary points by introducing an infinitesimal variation $\varepsilon \xi$ in the tangential direction:

$$\delta \xi E(\varphi) = \left( \varepsilon \int_M (d\xi \wedge \star d\varphi) + (d\varphi \wedge \star d\xi) \right) + o(\varepsilon^2) = -\varepsilon \int_M (\xi, d \star d\varphi) + o(\varepsilon^2)$$

so $\varphi$ is stationary (“harmonic”) if and only if $d \star d\varphi$ is totally normal – that is, a multiple of $\varphi$. This multiple is actually determined, since $|d\varphi|^2 = (d\varphi \wedge \star d\varphi) = d(\varphi, \star d\varphi) - (\varphi, d \star d\varphi) = -(\varphi, d \star d\varphi)$, so $\varphi : M \rightarrow S^2$ is harmonic if and only if it satisfies the equation

$$\triangle \varphi + |d\varphi|^2 \varphi = 0$$

Now suppose we have an arbitrary map $J : M \rightarrow S^2$, and take $S^2$ to be the unit imaginary quaternions. We may then define the operators $\partial_J, \bar{\partial}_J$ on maps from $M$ to $\mathbb{H}$ by

$$\partial_J = \frac{1}{2}(d - J \ast d)$$

$$\bar{\partial}_J = \frac{1}{2}(d + J \ast d)$$
splitting $d$ into $J$-commuting and $J$-anticommuting parts. If $f : M \rightarrow \mathbb{E}^3 = \text{im} \mathbb{H}$ then since $J^2 = -1$, the equation $\partial_J f = 0$ simply becomes the moving Cauchy-Riemann equations for $f$. Solutions correspond to branched conformally immersed surfaces in $\mathbb{E}^3$ with normal map given by $J$.

Now suppose $J$ is also harmonic, and define the $\text{im} \mathbb{H}$-valued 1-form $\omega = \partial_J J$. Then $\omega$ is closed:

$$d\omega = d \partial_J J = \frac{1}{2} (d^2 J - dJ \wedge \ast dJ - J \cdot d \ast dJ)$$

$$= \frac{1}{2} (|dJ|^2 + J \cdot |dJ|^2) \, dx \wedge dy$$

and furthermore, $\ast \omega = \ast \partial_J J = J \partial_J J = J \omega$, so $\omega$ has an antiderivative $f$ which is a map from $M$ to $\mathbb{E}^3$ and for which $J$ is the normal map.

Next, consider the mean curvature equation for such a conformal map to $\mathbb{E}^3$: $d \ast df = H \, df \wedge df$. If $f$ came from a harmonic $J$ then

$$d \ast df = d(J df) = df \wedge df + \partial_J J \wedge df$$

where we have used the decomposition $d = \partial_J + \bar{\partial}_J$. The second term in this sum is zero in order to make the two sides of the equation have matching type\(^1\), which immediately implies that the mean curvature is constant. In general this will give surfaces of constant mean curvature 1, but if $d \ast df = 0$ then the surfaces are actually minimal. This condition, that $\partial_J J \wedge \partial_J J = 0$, is simply the statement that $J : M \rightarrow S^2$ is holomorphic (or antiholomorphic).

Altogether, the above arguments show that: every surface of constant mean curvature in $\mathbb{E}^3$ can be obtained from a harmonic map $J : M \rightarrow S^2$, and such a surface is minimal if and only if $J$ is conformal.

**Duality between CMC surfaces in $\mathbb{E}^3$ and minimal surfaces in $S^3$:** As before, let $f : M \rightarrow \mathbb{E}^3 = \text{im} \mathbb{H}$ be a surface of constant mean curvature (either 1 or 0), so $f$ satisfies the equation $d \ast df = df \wedge df$. Conversely, any CMC surface on $M$ comes from integrating an $\text{im} \mathbb{H}$-valued 1-form $\omega$ satisfying the (linear) integrability equation

$$d\omega = 0$$

and the (nonlinear) CMC equation

$$d \ast \omega = \omega \wedge \omega$$

Now set $\eta = \ast \omega$. The CMC equation becomes $d\eta - \ast \eta \wedge \ast \eta = 0$, but $\ast \alpha \wedge \ast \beta = \alpha \wedge \beta$, so this becomes the nonlinear integrability equation

$$d\eta - \eta \wedge \eta = 0$$

\(^1\)In fact, the equation $\eta \wedge df = 0$ holds for $\text{im} \mathbb{H}$-valued 1-forms if and only if $\ast \eta = -J \eta$. Since this is automatically true when $\eta = \bar{\partial}_J g$, $\bar{\partial}_J J \wedge \bar{\partial}_J J = 0$. 

2
This is the right-invariant Maurer-Cartan equation for a map from $M$ to $SU(2) \cong S^3$. Such an $\eta$ can therefore be integrated to a function $F : \tilde{M} \rightarrow S^3 \subset \mathbb{H}$, unique up to constant right multiples by elements of $S^3$.

$S^3$ carries a natural nondegenerate metric via the Killing form $\kappa$, so we can again define the elastic energy $E(\varphi) = \frac{1}{2} \int_M d\varphi \wedge * d\varphi$ and the stationary points will be given by solutions to the harmonic equation $d * \varphi = 0$. Of course, since $d\omega = 0$ we have $d * \eta = 0$, so $\eta$ satisfies both the nonlinear integrability equation and the linear harmonicity equation for a map into $S^3$.

This means that every minimal surface in $S^3$ gives a CMC surface in $\mathbb{E}^3$ and vice versa.

**Harmonic Maps into $SU(2)$:** Let $f : M \rightarrow SU(2)$ be a map with Maurer-Cartan derivative $\omega = f^{-1} df \in \Omega^1(M; su(2))$. Then $f$ is a conformally parametrized harmonic map if and only if the equation $d* \omega = 0$ holds. Conversely, a $su(2)$-valued 1-form $\omega$ can be locally integrated to a harmonic map exactly when the two equations

$$d\omega + \frac{1}{2} [\omega \wedge \omega] = 0$$
$$d* \omega = 0$$

hold, and the integral is unique up to right translation. Now let $\lambda = \exp\{*\theta\}$ and define the loop of 1-forms

$$\omega_\lambda = \left(\frac{1 - \lambda}{2}\right) \omega = \left(\frac{1 - \cos \theta}{2}\right) \omega - \left(\frac{\sin \theta}{2}\right) * \omega$$

This integrates to a loop of $SU(2)$-valued maps exactly when

$$0 = d\omega_\lambda + \frac{1}{2} [\omega_\lambda \wedge \omega_\lambda]$$
$$= \left(\frac{1 - \cos \theta}{2}\right) d\omega - \left(\frac{\sin \theta}{2}\right) d* \omega + \left(\frac{1 - \cos \theta}{2}\right)^2 + \left(-\frac{\sin \theta}{2}\right)^2 \cdot \frac{1}{2} [\omega \wedge \omega]$$
$$= \left(\frac{1 - \cos \theta}{2}\right) \cdot \left(\frac{d\omega + \frac{1}{2} [\omega \wedge \omega]}{2} - \left(\frac{\sin \theta}{2}\right) \cdot d* \omega$$

where we have used $[* \omega \wedge * \omega] = [\omega \wedge \omega] = 0$. This demonstrates that $\omega$ integrates to a harmonic map if and only if the loop of forms $\omega_\lambda$ is integrable. Thus, the problem of finding a harmonic map is recast as a problem of finding a loop in $SU(2)$ with a certain dependence on $\lambda$. More explicitly, split $\omega$ by type: $\omega = \omega' + \omega'' \in \Omega^{(1,0)} \oplus \Omega^{(0,1)}$. Then $* \omega = i(\omega' - \omega'')$, so the dependency on $\lambda$ is of the form

$$\omega_\lambda = \left(\frac{1 - \lambda}{2}\right) (\omega' + \omega'') = \left(\frac{1 - \lambda}{2}\right) \omega' + \left(\frac{1 - \lambda^{-1}}{2}\right) \omega''$$

(Uhl)

This observation might fairly be called Uhlenbeck's lemma.
To work directly with these loops, we introduce the loop groups \( \Lambda_0 G = \{ g_\bullet : S^1 \to G \mid g_1 = 1 \} \) with Lie algebras \( \Lambda_0 \mathfrak{g} = \{ g_\bullet : S^1 \to \mathfrak{g} \mid g_1 = 0 \} \). These are infinite-dimensional Banach Lie groups/algebras.

The key tool will be the Iwasawa decomposition of \( \Lambda_0 SL(2, \mathbb{C}) \). Let \( \Lambda_0^+ SL(2, \mathbb{C}) \) denote the loops in \( SL(2, \mathbb{C}) \) parameterized by \( \lambda \) which extend analytically to the unit disc. These are a loop group analog of upper triangular matrices. The Iwasawa decomposition is the generalization to loop groups of the Hilbert-Schmidt decomposition:

\[
\Lambda_0 SL(2, \mathbb{C}) = \Lambda_0 SU(2) \cdot \Lambda_0^+ SL(2, \mathbb{C})
\]

This decomposition is particularly nice in the Lie algebra: Let \( \xi \in \Lambda_0 \mathfrak{sl}(2, \mathbb{C}) \) be given, and write the Fourier decomposition as a sum of negative and nonnegative parts:

\[
\xi = \sum_{i < 0} \xi_i \lambda^i + \sum_{j \geq 0} \xi_j \lambda^j = \xi^- + \xi^+
\]

where the coefficients \( \xi_k \) are in \( \mathfrak{sl}(2, \mathbb{C}) \) and subject to the condition \( \sum \xi_k = 0 \). The projection from \( \mathfrak{sl}(2, \mathbb{C}) \) to \( \mathfrak{su}(2) \) is given by \( \zeta \mapsto \zeta - \zeta^\dagger \), so we can take

\[
\xi = (\xi^- - \xi^\dagger) + (\xi^+ + \xi^\dagger) \in \Lambda_0 \mathfrak{su}(2) \oplus \Lambda_0^+ \mathfrak{sl}(2, \mathbb{C})
\]

It is nearly immediate from this decomposition that any form \( \omega \) solving \( (Uhl) \) will be given as the \( \Lambda_0 \mathfrak{su}(2) \)-part of a form

\[
\xi = \sum_{k=-1}^{\infty} \xi_k \lambda^k
\]

with \( \xi''_{-1} = 0 \). \( \xi \) is called the holomorphic potential.

**** (how to get meromorphic poten.? **** So we can actually obtain all solutions to \( (Uhl) \) by starting with a meromorphic potential

\[
\xi = \xi_{-1} \lambda^{-1} dz
\]

where \( \xi_{-1} \) is a meromorphic \( \mathfrak{sl}(2, \mathbb{C}) \)-valued function of \( z \).

Reversing our steps is straightforward. Given a meromorphic potential \( \xi \), integrate to a function \( C : \tilde{M} \to \Lambda_0 SU(2, \mathbb{C}) \) such that \( dC = \xi C \). Split \( C = A \cdot B \) using the Iwasawa decomposition, so \( A \in \Lambda_0 SL(2) \) and \( B \in \Lambda_0^+ SU(2, \mathbb{C}) \). Then \( A_{-1} : \tilde{M} \to SU(2) \) is the desired harmonic map.