Let M be a Riemann surface and identify $\mathbb{R}^3$ with the imaginary quaternions $\mathrm{im}\,\mathbb{H}$. Then an immersion $M \xrightarrow{f} \mathbb{R}^3$ with unit normal N is conformal if and only if it satisfies the "moving Cauchy-Riemann equations"

$$*\,df = N \cdot df$$

so that $f_y = N \cdot f_x$. Given such an f, the tangential part of the Laplacian vanishes:

$$(*\,d*df, df) = (*\,dN \wedge df, df) = 0$$

since dN is tangential and the quaternion product of two vectors is perpendicular to their span. In fact, we can be more precise: if $\lambda = |f_x|^2 = |f_y|^2$ is the scaling factor of f,

$$d*df = (f_{xx} + f_{yy})^{\perp} dx \wedge dy = -2\lambda H dx \wedge dy = -H\, df \wedge df$$

where H is the mean curvature of f. In particular, $f(M) \subset \mathbb{R}^3$ is a minimal surface if and only if f is harmonic, $d*df = 0$.

From a slightly different perspective, we have that f is minimal iff

$$0 = (\triangle f, N) = -\operatorname{tr}(df, dN) = -f_x N_x - f_y N_y = -f_x(N_x - NN_y)$$

This shows that $f(M)$ is a minimal surface exactly when the map $N : M \longrightarrow S^2$ is an antiholomorphic map.

Harmonic maps to S^2: Let $(,)$ denote the inner product on the round S^2, and define the elastic energy functional

$$E(\varphi) = \frac{1}{2}\int_M (d\varphi \wedge *d\varphi)$$

on maps $\varphi : M \longrightarrow S^2$. We investigate stationary points by introducing an infinitessimal variation $\varepsilon\xi$ in the tangential direction:

$$\delta_\xi E(\varphi) = \left(\frac{\varepsilon}{2}\int_M (d\xi \wedge *d\varphi) + (d\varphi \wedge *d\xi)\right) + o(\varepsilon^2) = -\varepsilon \int_M (\xi, d*d\varphi) + o(\varepsilon^2)$$

so φ is stationary ("*harmonic*") if and only if $d*d\varphi$ is totally normal – that is, a multiple of φ. This multiple is actually determined, since $|d\varphi|^2 = (d\varphi \wedge *d\varphi) = d(\varphi, *d\varphi) - (\varphi, d*d\varphi) = -(\varphi, d*d\varphi)$, so $\varphi : M \longrightarrow S^2$ is harmonic if and only if it satisfies the equation

$$\triangle\varphi + |d\varphi|^2\varphi = 0$$

Now suppose we have an arbitrary map $J : M \longrightarrow S^2$, and take S^2 to be the unit imaginary quaternions. We may then define the operators $\partial_J, \bar\partial_J$ on maps from M to $\mathbb{H}$ by

$$\partial_J = \frac{1}{2}(d - J*d)$$

$$\bar\partial_J = \frac{1}{2}(d + J*d)$$

splitting d into J-commuting and J-anticommuting parts. If $f : M \longrightarrow \mathbb{E}^3 = \operatorname{im}\mathbb{H}$ then since $J^2 = -1$, the equation $\bar{\partial}_J f = 0$ simply becomes the moving Cauchy-Riemann equations for f. Solutions correspond to branched conformally immersed surfaces in $\mathbb{E}^3$ with normal map given by J.

Now suppose J is also harmonic, and define the $\operatorname{im}\mathbb{H}$-valued 1-form $\omega = \partial_J J$. Then ω is closed:

$$\begin{aligned} d\omega = d\,\partial_J J &= \frac{1}{2}\left(d^2 J - dJ \wedge *dJ - J \cdot d * dJ\right) \\ &= \frac{1}{2}\left(|dJ|^2 + J \cdot |dJ|^2 J\right)\, dx \wedge dy \\ &= \frac{1}{2}\left(|dJ|^2 - |dJ|^2\right)\, dx \wedge dy = 0 \end{aligned}$$

and furthermore, $*\omega = *\partial_J J = J\partial_J J = J\omega$, so ω has an antiderivative f which is a map from $\tilde{M}$ to $\mathbb{E}^3$ and for which J is the normal map.

Next, consider the mean curvature equation for such a conformal map to $\mathbb{E}^3$: $d * df = H\, df \wedge df$. If f came from a harmonic J then

$$d * df = d(Jdf) = df \wedge df + \bar{\partial}_J J \wedge df$$

where we have used the decomposition $d = \partial_J + \bar{\partial}_J$. The second term in this sum is zero in order to make the two sides of the equation have matching type[1], which immediately implies that the mean curvature is constant. In general this will give surfaces of constant mean curvature 1, but if $d * df = 0$ then the surfaces are actually minimal. This condition, that $\partial_J J \wedge \partial_J J = 0$, is simply the statement that $J : M \longrightarrow S^2$ is holomorphic (or antiholomorphic).

Altogether, the above arguments show that: every surface of constant mean curvature in $\mathbb{E}^3$ can be obtained from a harmonic map $J : M \longrightarrow S^2$, and such a surface is minimal if and only if J is conformal.

Duality between CMC surfaces in $\mathbb{E}^3$ and minimal surfaces in S^3: As before, let $f : M \longrightarrow \mathbb{E}^3 = \operatorname{im}\mathbb{H}$ be a surface of constant mean curvature (either 1 or 0), so f satisfies the equation $d * df = df \wedge df$. Conversely, any CMC surface on $\tilde{M}$ comes from integrating an $\operatorname{im}\mathbb{H}$-valued 1-form ω satisfying the (linear) integrability equation

$$d\omega = 0$$

and the (nonlinear) CMC equation

$$d * \omega = \omega \wedge \omega$$

Now set $\eta = *\omega$. The CMC equation becomes $d\eta - *\eta \wedge *\eta = 0$, but $*\alpha \wedge *\beta = \alpha \wedge \beta$, so this becomes the nonlinear integrability equation

$$d\eta - \eta \wedge \eta = 0$$

[1] In fact, the equation $\eta \wedge df = 0$ holds for $\operatorname{im}\mathbb{H}$-valued 1-forms if and only if $*\eta = -J\eta$. Since this is automatically true when $\eta = \bar{\partial}_J g$, $\bar{\partial}_J J \wedge \partial_J J = 0$.

This is the right-invariant Maurer-Cartan equation for a map from M to $SU(2) \cong S^3$. Such an η can therefore be integrated to a function $F : \tilde{M} \longrightarrow S^3 \subset \mathbb{H}$, unique up to constant right multiples by elements of S^3.

S^3 carries a natural nondegenerate metric via the Killing form κ, so we can again define the elastic energy $E(\varphi) = \frac{1}{2}\int_M d\varphi \wedge_\kappa * d\varphi$ and the stationary points will be given by solutions to the harmonic equation $d * \varphi = 0$. Of course, since $d\omega = 0$ we have $d * \eta = 0$, so η satisfies both the nonlinear integrability equation and the linear harmonicity equation for a map into S^3.

This means that every minimal surface in S^3 gives a CMC surface in $\mathbb{E}^3$ and vice versa.

Harmonic Maps into $\mathsf{SU}(2)$*:* Let $f : M \longrightarrow \mathsf{SU}(2)$ be a map with Maurer-Cartan derivative $\omega = f^{-1}df \in \Omega^1(M; \mathfrak{su}(2))$. Then f is a conformally parametrized harmonic map if and only if the equation $d * \omega = 0$ holds. Conversely, a $\mathfrak{su}(2)$-valued 1-form ω can be locally integrated to a harmonic map exactly when the two equations

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = 0$$

$$d * \omega = 0$$

hold, and the integral is unique up to right translation. Now let $\lambda = \exp\{*\theta\}$ and define the loop of 1-forms

$$\omega_\lambda = \left(\frac{1-\lambda}{2}\right)\omega = \left(\frac{1-\cos\theta}{2}\right)\omega - \left(\frac{\sin\theta}{2}\right) * \omega$$

This integrates to a loop of $\mathsf{SU}(2)$-valued maps exactly when

$$\begin{aligned} 0 &= d\omega_\lambda + \frac{1}{2}[\omega_\lambda \wedge \omega_\lambda] \\ &= \left(\frac{1-\cos\theta}{2}\right) d\omega - \left(\frac{\sin\theta}{2}\right) d * \omega + \left(\left(\frac{1-\cos\theta}{2}\right)^2 + \left(\frac{-\sin\theta}{2}\right)^2\right) \cdot \frac{1}{2}[\omega \wedge \omega] \\ &= \left(\frac{1-\cos\theta}{2}\right) \cdot \left(d\omega + \frac{1}{2}[\omega \wedge \omega]\right) - \left(\frac{\sin\theta}{2}\right) \cdot d * \omega \end{aligned}$$

where we have used $[*\omega \wedge *\omega] = [\omega \wedge \omega]$ and $[\omega \wedge *\omega] = 0$. This demonstrates that ω integrates to a harmonic map if and only if the loop of forms ω_λ is integrable. Thus, the problem of finding a harmonic map is recast as a problem of finding a loop in $\mathsf{SU}(2)$ with a certain dependence on λ. More explicitly, split ω by type: $\omega = \omega' + \omega'' \in \Omega^{(1,0)} \oplus \Omega^{(0,1)}$. Then $*\omega = i(\omega' - \omega'')$, so the dependency on λ is of the form

$$\omega_\lambda = \left(\frac{1-\lambda}{2}\right)(\omega' + \omega'') = \left(\frac{1-\lambda}{2}\right)\omega' + \left(\frac{1-\lambda^{-1}}{2}\right)\omega'' \tag{Uhl}$$

This observation might fairly be called Uhlenbeck's lemma.

To work directly with these loops, we introduce the loop groups $\Lambda_0 G = \{g_\bullet : S^1 \longrightarrow G \mid g_1 = 1\}$ with Lie algebras $\Lambda_0\mathfrak{g} = \{g_\bullet : S^1 \longrightarrow \mathfrak{g} \mid g_1 = 0\}$. These are infinite-dimensional Banach Lie groups/algebras.

The key tool will be the Iwasawa decomposition of $\Lambda_0\mathsf{SL}(2,\mathbb{C})$. Let $\Lambda_0^+\mathsf{SL}(2,\mathbb{C})$ denote the loops in $\mathsf{SL}(2,\mathbb{C})$ parameterized by λ which extend analytically to the unit disc. These are a loop group analog of upper triangular matrices. The Iwasawa decomposition is the generalization to loop groups of the Hilbert-Schmidt decomposition:

$$\Lambda_0\mathsf{SL}(2,\mathbb{C}) = \Lambda_0\mathsf{SU}(2) \cdot \Lambda_0^+\mathsf{SL}(2,\mathbb{C})$$

This decomposition is particularly nice in the Lie algebra: Let $\xi \in \Lambda_0\mathfrak{sl}(2,\mathbb{C})$ be given, and write the Fourier decomposition as a sum of negative and nonnegative parts:

$$\xi = \sum_{i<0} \xi_i \lambda^i + \sum_{j \geq 0} \xi_j \lambda^j = \xi_- + \xi_+$$

where the coefficients ξ_k are in $\mathfrak{sl}(2,\mathbb{C})$ and subject to the condition $\sum \xi_k = 0$. The projection from $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{su}(2)$ is given by $\zeta \mapsto \zeta - \zeta^\dagger$, so we can take

$$\xi = (\xi_- - \xi_-^\dagger) + (\xi_+ + \xi_-^\dagger) \in \Lambda_0\mathfrak{su}(2) \oplus \Lambda_0^+\mathfrak{sl}(2,\mathbb{C})$$

It is nearly immediate from this decomposition that any form ω solving (Uhl) will be given as the $\Lambda_0\mathfrak{su}(2)$-part of a form

$$\xi = \sum_{k=-1}^{\infty} \xi_k \lambda^k$$

with $\xi_{-1}'' = 0$. ξ is called the holomorphic potential.

**** (how to get meromorphic poten.? **** So we can actually obtain all solutions to (Uhl) by starting with a meromorphic potential

$$\xi = \xi_{-1}\lambda^{-1} dz$$

where ξ_{-1} is a meromorphic $\mathfrak{sl}(2,\mathbb{C})$-valued function of z.

Reversing our steps is straightforward. Given a meromorphic potential ξ, integrate to a function $C : \tilde{M} \longrightarrow \Lambda_0\mathsf{SU}(2,\mathbb{C})$ such that $dC = \xi C$. Split $C = A \cdot B$ using the Iwasawa decomposition, so $A \in \Lambda_0\mathsf{SL}(2)$ and $B \in \Lambda_0^+\mathsf{SU}(2,\mathbb{C})$. Then $A_{-1} : \tilde{M} \longrightarrow \mathsf{SU}(2)$ is the desired harmonic map.