Due in class on Thursday, September 27.

1. **Do not do this problem, it is wrong. I’m leaving it here to record my wrongness and to keep the numbers consistent.**

The (open) \( n \)-dimensional simplex is the set \( \Delta^n \subset [0,1]^n \) defined by

\[
\Delta^n = \{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 < \cdots < x_n \}.
\]

For example, \( \Delta^2 \) is a triangle, \( \Delta^3 \) is a tetrahedron, and so on.

Let \( m_n \) denote \( n \)-dimensional Lebesgue measure on \([0,1]^n\).

(a) (Warmup) Show that \( m_n(\Delta^n) = \frac{1}{n!} \). (Induction may be helpful.)

(b) Let \( \mu_n \) be the probability measure on \([0,1]^n\) that spreads a unit of mass uniformly on \( \Delta^n \), i.e.

\[ \mu_n(A) = n! m_n(A \cap \Delta^n) \]. Show that the sequence \( \{\mu_n\} \) is consistent in the sense of the Kolmogorov extension theorem.

(c) Let \( \mu \) be the limiting measure on \([0,1]^N\) produced by applying the Kolmogorov extension theorem to \( \{\mu_n\} \). If \( \Delta \subset [0,1]^N \) is the set of all strictly increasing sequences which converge to 1, show that \( \mu(\Delta) = 1 \).

(d) On the other hand, if \( m \) is Lebesgue measure on \([0,1]^N\) (i.e. the limiting measure of \( \{m_n\} \)), show that \( m(\Delta) = 0 \).

(e) (Bonus problem) Suppose \( U_1, U_2, \ldots \) is an iid sequence of uniform \((0,1)\) random variables on some probability space. Use the \( U_i \) to directly construct a sequence of random variables \( X_1, X_2, \ldots \) whose joint distribution is \( \mu \).

2. For each of the following sequences of probability measures on \( \mathbb{R} \), determine whether the sequence converges weakly, and if so find its weak limit. \( m \) denotes Lebesgue measure and \( \delta_x \) is the Dirac delta measure at \( x \) (i.e. \( \delta_x(A) = 1 \) if \( x \in A \) and \( 0 \) if \( x \notin A \)).

Recall the definition: \( \mu_n \to \mu \) weakly iff for every bounded continuous \( f \), we have

\[ \int f \, d\mu_n \to \int f \, d\mu. \]

(a) \( \mu_n \) is uniform measure on \([0,1/n]\) (i.e. \( \mu_n(A) = n \cdot m(A \cap [0,1/n]) \)).

(b) \( \mu_n = \delta_n \).

(c) \( \mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{1/i} \).

(d) (Bonus problem) \( \mu_n \) is uniform measure on \([0,n]\), i.e. \( \mu_n(A) = \frac{1}{n} m(A \cap [0,n]) \). (This is maybe a bit tricky to prove. Spend a little time thinking about it but don’t waste your entire week.)

3. (a) Suppose \( X_1, X_2, \ldots, X \) are random variables, and \( X_n \to X \) in probability. Show that \( X_n \to X \) weakly, i.e. if \( \mu_n, \mu \) are the distributions of \( X_n, X \), then \( \mu_n \to \mu \) weakly.

(b) Suppose \( X_1, X_2, \ldots \) are random variables, \( X_n \sim \mu_n \), and \( \mu_n \to \delta_c \) weakly. Show that \( X_n \to c \) in probability.