1. Let $X$ be a random variable. Prove that the following are equivalent:
   (a) $X$ is independent of every random variable $Y$.
   (b) $X$ is independent of itself.
   (c) For all events $A \in \sigma(X)$, $P(A) = 0$ or $P(A) = 1$. (We say the $\sigma$-field $\sigma(X)$ is almost trivial.)
   (d) There exists a constant $c \in \mathbb{R}$ such that $X = c$ a.s.

2. (Like Durrett 2.1.4) Let $g_1, \ldots, g_n$ be probability density functions (i.e. $g_i : \mathbb{R} \to [0, \infty)$ is measurable and $\int_{\mathbb{R}} g_i \, dm = 1$), and define $f : \mathbb{R}^n \to [0, \infty)$ by $f(x_1, \ldots, x_n) = g_1(x_1) \cdots g_n(x_n)$. Show that $X = (X_1, \ldots, X_n)$ is a random vector with density $f$ if and only if $X_1, \ldots, X_n$ are independent random variables where $X_i$ has density $g_i$.
   (Remark: In Durrett’s version, the $g_i$ are not assumed to be probability densities, i.e. it is not assumed that $\int_{\mathbb{R}} g_i \, dm = 1$. However, one can just rescale them to achieve this.)

3. (a) Let $\mathcal{G}$ be a $\sigma$-field, and let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots$ be an increasing sequence of $\sigma$-fields. Suppose that for each $n$, $\mathcal{G}$ and $\mathcal{G}_n$ are independent. Let $\mathcal{G}_\infty = \sigma(\mathcal{G}_1, \mathcal{G}_2, \ldots)$ be the $\sigma$-field generated by the $\mathcal{G}_n$ (i.e. the smallest $\sigma$-field such that $\mathcal{G}_n \subset \mathcal{G}_\infty$ for all $n$). Show that $\mathcal{G}$ and $\mathcal{G}_\infty$ are independent. (Hint: $\bigcup_{n=1}^\infty \mathcal{G}_n$ is a $\pi$-system.)
   (b) Let $Y$ be a random variable, and $X_1, X_2, \ldots$ a sequence of random variables such that $Y$ is independent of $(X_1, \ldots, X_n)$ for each $n$. Show that $Y$ is independent of $\sup_n X_n$. (It is also independent of $\inf_n X_n$, $\lim \sup_n X_n$, etc.) (Hint: Use part (a).)

4. (Durrett 2.1.13) Show that if $X, Y$ are independent discrete random variables, then
   $$P(X + Y = n) = \sum_m P(X = m)P(Y = n - m)$$

5. (Durrett 2.1.14) Recall that a random variable $Z$ has the Poisson distribution with parameter $\lambda$ if $P(Z = k) = e^{-\lambda} \lambda^k / k!$ for $k = 0, 1, 2, \ldots$; we write $Z \sim \text{Poisson}(\lambda)$. Suppose $X, Y$ are independent with $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$. Use the previous exercise to show that $X + Y \sim \text{Poisson}(\lambda + \mu)$. 

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Due in class on Thursday, September 13.

1. Let $X$ be a random variable. Prove that the following are equivalent:
   (a) $X$ is independent of every random variable $Y$.
   (b) $X$ is independent of itself.
   (c) For all events $A \in \sigma(X)$, $P(A) = 0$ or $P(A) = 1$. (We say the $\sigma$-field $\sigma(X)$ is almost trivial.)
   (d) There exists a constant $c \in \mathbb{R}$ such that $X = c$ a.s.

2. (Like Durrett 2.1.4) Let $g_1, \ldots, g_n$ be probability density functions (i.e. $g_i : \mathbb{R} \to [0, \infty)$ is measurable and $\int_{\mathbb{R}} g_i \, dm = 1$), and define $f : \mathbb{R}^n \to [0, \infty)$ by $f(x_1, \ldots, x_n) = g_1(x_1) \cdots g_n(x_n)$. Show that $X = (X_1, \ldots, X_n)$ is a random vector with density $f$ if and only if $X_1, \ldots, X_n$ are independent random variables where $X_i$ has density $g_i$.
   (Remark: In Durrett’s version, the $g_i$ are not assumed to be probability densities, i.e. it is not assumed that $\int_{\mathbb{R}} g_i \, dm = 1$. However, one can just rescale them to achieve this.)

3. (a) Let $\mathcal{G}$ be a $\sigma$-field, and let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots$ be an increasing sequence of $\sigma$-fields. Suppose that for each $n$, $\mathcal{G}$ and $\mathcal{G}_n$ are independent. Let $\mathcal{G}_\infty = \sigma(\mathcal{G}_1, \mathcal{G}_2, \ldots)$ be the $\sigma$-field generated by the $\mathcal{G}_n$ (i.e. the smallest $\sigma$-field such that $\mathcal{G}_n \subset \mathcal{G}_\infty$ for all $n$). Show that $\mathcal{G}$ and $\mathcal{G}_\infty$ are independent. (Hint: $\bigcup_{n=1}^\infty \mathcal{G}_n$ is a $\pi$-system.)
   (b) Let $Y$ be a random variable, and $X_1, X_2, \ldots$ a sequence of random variables such that $Y$ is independent of $(X_1, \ldots, X_n)$ for each $n$. Show that $Y$ is independent of $\sup_n X_n$. (It is also independent of $\inf_n X_n$, $\lim \sup_n X_n$, etc.) (Hint: Use part (a).)

4. (Durrett 2.1.13) Show that if $X, Y$ are independent discrete random variables, then
   $$P(X + Y = n) = \sum_m P(X = m)P(Y = n - m)$$

5. (Durrett 2.1.14) Recall that a random variable $Z$ has the Poisson distribution with parameter $\lambda$ if $P(Z = k) = e^{-\lambda} \lambda^k / k!$ for $k = 0, 1, 2, \ldots$; we write $Z \sim \text{Poisson}(\lambda)$. Suppose $X, Y$ are independent with $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$. Use the previous exercise to show that $X + Y \sim \text{Poisson}(\lambda + \mu)$. 