Due in class on Thursday, September 6.

1. (Durrett 1.6.1) Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is strictly convex, i.e.

$$\varphi(tx + (1-t)y) < t\varphi(x) + (1-t)\varphi(y)$$

for all $x \neq y$ and $0 < t < 1$. (“Convex” only requires $\leq$ in the above inequality.) Show that under this assumption, equality holds in Jensen’s inequality only in the trivial case that $X$ is a.s. constant. That is, if $X$ and $\varphi(X)$ are integrable and $E[\varphi(X)] = \varphi(EX)$ then $X = EX$ a.s.

2. Suppose $X_n \to X$ in probability and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that $f(X_n) \to f(X)$ in probability.

(Durrett has a proof at Theorem 2.3.4 using the “double subsequence” trick, but for this problem, please prove it directly from the definition of convergence i.p. Hint: Break up the event \{|$f(X_n) - f(X)$| > $\epsilon$\} according to whether $|X| \leq M$ or $|X| > M$ for some large $M$. Also, remember that $f$ is uniformly continuous on compact intervals.)

3. Suppose $X_n \to X$ in probability. Show that, almost surely,

$$\liminf_{n \to \infty} X_n \leq X \leq \limsup_{n \to \infty} X_n.$$  

(Either work directly or use the double subsequence trick.)

4. A set $S$ of random variables is said to be uniformly integrable or ui if for every $\epsilon > 0$ there exists $M > 0$ such that for all $X \in S$,

$$E[|X|1_{\{|X| \geq M\}}] < \epsilon.$$  

Prove the “crystal ball condition”: Let $S$ be a set of random variables. If for some $p > 1$ we have $\sup_{X \in S} E[|X|^p] < \infty$ (i.e. $S$ is bounded in $L^p$ norm) then $S$ is uniformly integrable.

5. Let $S, S'$ be two sets of random variables. Suppose that $S'$ is ui, and for every $X \in S$ there exists $Y \in S'$ with $|X| \leq |Y|$ a.s. Show that $S$ is also ui.

(In particular, the Vitali convergence theorem implies the dominated convergence theorem. Note, however, that we used dominated convergence in our proof of Vitali.)