Due in class on Friday, December 3.

1. Let $X$ be a random variable taking integer values, and let $\varphi$ be its characteristic function.
   
   (a) Show that $\varphi$ is $2\pi$-periodic, i.e. $\varphi(t + 2\pi) = \varphi(t)$ for all $t$.
   (b) Use the previous part to show that $\int_{-\infty}^{\infty} |\varphi(t)| \, dt = \infty$.
   (c) The Fourier inversion formula we proved in class does not apply to $X$, since $\varphi$ is not integrable
       and $X$ does not have a density. However, show that for any $k \in \mathbb{Z}$, we have
       
       \[
P(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \varphi(t) \, dt.
       \]

2. Let $X_1, X_2, \ldots$ be a sequence of random variables with chfs $\varphi_1, \varphi_2, \ldots$, and let $p_1, p_2, \ldots$ be a sequence
   in $[0, 1]$ with $\sum_{k=1}^{\infty} p_k = 1$. Show that $\varphi(t) = \sum_{k=1}^{\infty} p_k \varphi_k(t)$ is a chf, by exhibiting a random variable
   $X$ whose chf it is.

3. A function $\psi : \mathbb{R} \to \mathbb{C}$ is called non-negative definite if for any $t_1, \ldots, t_n \in \mathbb{R}$ and any $c_1, \ldots, c_n \in \mathbb{C}$,
   we have
   
   \[
   \sum_{r,s=1}^{n} c_r \overline{c_s} \psi(t_r - t_s) \geq 0.
   \]
   (This says that if we let $a_{rs} = \psi(t_r - t_s)$, then the $n \times n$ complex matrix $(a_{rs})$ is non-negative definite.)
   Show that any characteristic function $\varphi$ is non-negative definite.
   
   This fact has a converse: every continuous, non-negative definite function $\psi : \mathbb{R} \to \mathbb{C}$ with $\psi(0) = 1$ is
   the characteristic function of some random variable. This is called Bochner’s theorem, and it tells
   us precisely which functions are chfs.

4. (Durrett 3.4.5) Let $X_1, X_2, \ldots$ be iid with mean 0 and variance $\sigma^2 \in (0, \infty)$. Let $S_n = X_1 + \cdots + X_n$, and let $Q_n = X_1^2 + \cdots + X_n^2$. Show that $S_n / \sqrt{Q_n} \Rightarrow N(0, 1)$.

5. Suppose $X, Y$ are iid with mean 0 and variance 1. Show that $X, Y$ are $N(0, 1)$ iff $\frac{X + Y}{\sqrt{2}} \overset{d}{=} X \overset{d}{=} Y$. (Try
   using chfs for one direction, and the central limit theorem for the other.)

6. Let $X_1, X_2, \ldots$ be iid with mean $\mu$ and variance $\sigma^2 \in (0, \infty)$. Let $\bar{X}_n = \frac{1}{n} (X_1 + \cdots + X_n)$ (statisticians
   call this the sample mean). Let $g : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at $\mu$ and with
   $g' (\mu) \neq 0$. Show that:
   
   \[
   \sqrt{n} \left( g(\bar{X}_n) - g(\mu) \over \sigma g'(\mu) \right) \Rightarrow N(0, 1).
   \]
   
   In other words, the distribution of $g(\bar{X}_n)$ is approximately $N( g(\mu), \sigma^2 g'(\mu)^2 / n)$. Notice that $g(x) = x$
   is the central limit theorem. This establishes that not only is $\bar{X}_n$ approximately normally distributed
   for large $n$ (“asymptotically normal”), but so is any reasonable function of it. For reasons which I
   have never understood, statisticians call this fact the delta method.