Math 649

Homework 3
Due October 5, 2006

13. Prove by a direct computation, that
\[ e_{11} + e_{22} \quad \text{and} \quad e_{11}^2 + e_{12}e_{21} + e_{21}e_{12} + e_{22}^2 \]
belong to the center of the universal Lie algebra \( U(\mathcal{L}(2)) \) over \( \mathcal{L}(2) \).  

14. Choose one of two versions of this problem:
(a) Find, by direct computation, all differential operators of the second order in \( \mathbb{R}^n \) invariant with respect to all parallel translations and rotations.
(b) By referring to I. M. Gelfand’s note “Center of the infinitesimal group ring” prove that any differential operator in \( \mathbb{R}^n \) invariant with respect to all parallel translations and rotations has the form
\[ \sum_{k=0}^{m} a_k \Delta^k \]
where \( a_k \) are constants and \( \Delta \) is the Laplacian.

15. Let \((G, M)\) be a homogeneous space of a Lie group \( G \). Prove that, if a stabilizer of \( G \) is compact, then there exists a \( G \)-invariant Riemannian metric on \( M \).

16. Prove that the Euclidean 3-dimensional space with the cross product \( x \times y \) (introduced in elementary courses) is isomorphic to the Lie algebra of \( O(3) \).

17. Let \( \mathcal{A} \) be the group of all automorphisms of a Lie algebra \( L \). Prove that the set \( \mathfrak{A} \) of all derivations \( \varphi \) in \( L \) with \( [\varphi, \psi] = \varphi \psi - \psi \varphi \) is the Lie algebra of the group \( \mathcal{A} \).

18. To every \( a \in \mathcal{G} \) there corresponds an inner derivation \( f_a(x) = [a, x] \). Give an example of a derivation in a Lie algebra \( L \) which is not an inner derivation.

19. Prove that \( f(A) = -A^T \) is an automorphism of the Lie algebra \( \mathcal{L}(n) \) of \( n \times n \) matrices but not an inner automorphism.

20. If \( G \) is a linear group in a vector space \( E \) and if \( G \) is the Lie algebra of \( G \), then a bilinear form \( Q(x, y) \) is \( G \)-invariant if and only if \( Q(Ax, y) + Q(x, Ay) = 0 \) for all \( x, y \in E \).

21. Prove that the Lie algebra \( TR(n) \) of all \( n \times n \) upper triangular matrices is solvable by establishing that \( L_k = \{ A \in TR(n) : a_{ij} = 0 \quad \text{for} \quad i + k > j \}, \quad k = 0, 1, 2, \ldots \) are ideals and the quotient algebras \( L_k/L_{k-1} \) are abelian.

22. Prove that the Lie algebra of all \( n \times n \) complex matrices with trace 0 (which corresponds to the group \( SL(n, \mathbb{C}) \)) is semisimple by demonstrating that it has no commutative ideals besides \( \{0\} \).

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1. \( e_{ij} \) is a matrix with the entries \( a_{i'j'} = 1 \) for \( (i'j') = (ij) \) and the rest of the entries equal to 0.

2. A linear operator \( f \) in \( L \) is an isomorphism if \( f[x, y] = [f(x), f(y)] \) for all \( x, y \) and it is a derivation if \( f[x, y] = [f(x), y] + [x, f(y)] \) for all \( x, y \).

3. Inner automorphisms of \( \mathcal{L}(n) \) are defined by the formula \( f(A) = CAC^{-1} \) where \( C \in L(n) \).

4. A matrix \( (a_{ij}) \) belongs to \( TR(n) \) if \( a_{ij} = 0 \) for all \( i > j \).