6. Prove that the 3-dimensional Euclidean space $\mathbb{R}^3$ with the cross product $x \times y$ is a Lie algebra and that this algebra is isomorphic to the algebra of skew-symmetric $3 \times 3$ matrices.

Hint. If $e_1, e_2, e_3$ is an orthonormal basis in $\mathbb{R}^3$, then

$$e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2.$$ 

Find 3 linearly independent skew-symmetric matrices $E_1, E_2, E_3$ such that

$$E_1E_2 - E_2E_1 = E_3, E_2E_3 - E_3E_2 = E_1, E_3E_1 - E_1E_3 = E_2.$$ 

7. Let $A_t, t \in \mathbb{R}$ be a one-parameter group of $n \times n$ matrices (which means $A_sA_t = A_{s+t}$ for all $s, t$) such that $A_t \to A_0$ as $t \to 0$. Prove that $A_t = e^{tX}$ for some $X$. [Consider $B_t = \log A_t$.]

8. Prove that the Campbell-Hausdorff series converges if $\|X\|, \|Y\| < \delta$ for some $\delta > 0$. Find a $\delta$ for which this is true.

9. Let $P_n(x, y)$ be the sum of terms of degree $m$ in the Campbell-Hausdorff series. Deduce from Dynkin's formula for $P_n$ that $P_1 = x + y, P_2 = \frac{1}{2}[x, y]$ and find a similar expression for $P_3$.

By using these expressions, prove that, for every $x, y$,

$$\lim_{t \to 0} \frac{1}{t^2} e^{tx} e^{ty} e^{-tx} e^{-ty} = [x, y].$$

10. For which $k$ the system of differential operators $D, xD, \ldots, x^kD$ in $\mathbb{R}$ is complete?

Hint. It could be helpful to have an expression for $[x^iD, x^jD]$ for all $i, j$.

11. Describe the transformation groups in $\mathbb{R}$ corresponding to the Lie algebras with the basis:

(a) $D$; (b) $D, xD$; (c) $D, xD, x^2D$.

12. Compute the Lie algebra corresponding to the group of all transformations in $\mathbb{R}^3$ preserving the Euclidean metric.

Hint. Find the generators for the one-parameter groups of translations along the coordinate axes and rotations about these axes.