Math 649
Homework 1
Due September 7, 2006

1. Clearly, $L(n, R)$ is a subgroup of $L(n, C)$. Show that $L(n, C)$ can be considered as a subgroup of $L(2n, R)$.

2. Find a homomorphism $f$ from $SU(2)$ onto $SO(3)$ such that $f(x) = f(y)$ if and only if $x = \pm y$.
   
   *Hint.* Consider the space $V$ of $2 \times 2$ complex matrices $a$ such that $a^* = -a$ and $\text{tr}(a) = 0$ and put $f(x)(a) = xax^*$ for $x \in SU(2)$.

3. Groups $L(n, R), SL(n, R), O(n), SO(n)$ are manifolds in the $n^2$-dimensional linear space of $n \times n$ matrices with real entries. Find the tangent spaces to these manifold at the identity matrix $I = (\delta_{ij})$.

4. The exponential function of a matrix $x$ is defined by the series
   
   $$e^x = \sum_{0}^{\infty} \frac{x^n}{n!}.$$ 

   Find the diagonal entries of $e^x$ if $x$ is triangular (i.e., if $x_{ij} = 0$ for $i < j$). Use this to prove that $\text{det}(e^x) = e^{\text{tr}(x)}$.

5. There are many ways to define $|x|$ for a $n \times n$ matrix $x$ to satisfy conditions: $|x + y| \leq |x| + |y|, |cx| = |c||x|$ for $c \in R$ and $|xy| \leq |x||y|$. Describe some of such norms. Prove that, if $|x - I| < 1$, then
   
   $$y = \sum_{1}^{\infty} \frac{(-1)^{n-1}}{n} (x - I)^n$$

   if and only if $x = e^y$.  