1. Let $a_n = \sqrt{n}$.

(a) (5pts) Prove that for every $\varepsilon > 0$ there is $N \geq 1$ such that $|a_{n+1} - a_n| < \varepsilon$ if $n \geq N$.

**Solution:** Let $\varepsilon > 0$. Then $|a_{n+1} - a_n| < \varepsilon$ is equivalent with $|\sqrt{n+1} - \sqrt{n}| < \varepsilon$. Simplifying, this inequality is equivalent with

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \varepsilon.$$ 

Note that the previous expression is positive so we do not need the absolute value bars. Since

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$$

if we choose $N = (1/(2\varepsilon))^2$, it follows that $|a_{n+1} - a_n| < \varepsilon$ if $n \geq N$.

(b) (5pts) Is $a_n$ a Cauchy sequence? Why?

**Solution:** $a_n$ is not a Cauchy sequence. To prove this, suppose by contradiction that $a_n$ is Cauchy. Then $a_n$ must be convergent by Theorem 6.4. The sequence $a_n$ is, however, divergent. Thus it can not be Cauchy.
(7pts) 2. (a) Show that if $\sum a_n$ converges absolutely, then so does $\sum a_n^2$. Is this true without the hypothesis of absolute convergence (prove or give a counterexample)?

**Solution:** If the series $\sum a_n$ converges absolutely it converges. Then $\lim_{n \to \infty} a_n = 0$. Then there is $N \geq 1$ such that $|a_n| < 1$ for all $n \geq N$. It follows that $a_n^2 \leq |a_n|$ for all $n \geq N$. The comparison test for positive series implies that $\sum_{n=N}^{\infty} a_n^2$ converges. From the tail theorem we conclude that $\sum a_n^2$ converges.

The conclusion fails without the hypothesis of absolute convergence. Consider $\sum (-1)^n \sqrt{n}$. This series is convergent by the Cauchy test. The series $\sum \frac{1}{n}$, however, diverges (it is a $p$-series with $p = 1$).

(3pts) (b) If $\sum a_n$ converges and $a_n \geq 0$, does it follow that $\sum \sqrt{a_n}$ converges? Prove or give a counterexample.

**Solution:** If $\sum a_n$ converges and $a_n \geq 0$, it does not follow that $\sum \sqrt{a_n}$ converges. For example, $\sum \frac{1}{n^2}$ converges ($p$-series with $p = 2$) but $\sum \frac{1}{n}$ diverges.
(20pts) 3. Find the radius of convergence of \( \sum_{n=0}^{\infty} (\sin n)x^n \) (with proof).

**Solution:** The main point about this problem is that we cannot use the root or the ratio test to determine the radius of convergence of this series (because the sequences \((\sin(n))^{1/n}\) and \(\frac{\sin(n+1)}{\sin(n)}\) are divergent).

Notice that \(|\sin(n)x^n| \leq x^n\) for all \(n \geq 1\). If \(|x| < 1\) we know that \(\sum x^n\) converges absolutely. The comparison theorem implies that \(\sum \sin(n)x^n\) converges absolutely. For \(x = 1\), \(\sum \sin(n)\) diverges since \(\{\sin(n)\}\) does not converge to 0. Thus \(R = \) by Theorem-Definition 8.1.
(10pts) 4. Determine if the series
\[ \sum_{n=1}^{\infty} \ln \frac{n}{n+2} \]
is convergent. If yes, find it’s value.

**Solution:** Since \( \ln \frac{n}{n+2} = \ln(n) - \ln(n+2) \) we see that the series is a telescoping series. We compute the \( n \)th partial sum as follows:

\[
s_n = \sum_{k=1}^{n} (\ln(k) - \ln(k+2)) = \ln(1) + \ln(2) - \ln(n+1) - \ln(n+2).
\]

Then

\[
\lim_{n \to \infty} s_n = \ln(2) - \lim_{n \to \infty} (\ln(n+1) + \ln(n+2)) = -\infty.
\]

Thus the series is divergent.
5. Let \( b_n \) be a decreasing sequence with \( \lim_{n \to \infty} b_n = 0 \).

(a) Prove that \( \sum_{n=1}^{\infty} (b_n - b_{n+1}) \) converges.

**Solution:** We notice this is a telescoping series. The partial sum equals

\[
s_n = (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_n - b_{n+1}) = b_1 - b_{n+1}.
\]

Since \( \lim_{n \to \infty} b_n = 0 \) it follows that the sequence of partial sums is convergent and equals \( b_1 \).

Thus \( \sum (b_n - b_{n+1}) \) converges.

(b) Let \( a_n \) be a bounded sequence. Prove that \( \sum_{n=1}^{\infty} a_n(b_n - b_{n+1}) \) converges.

**Solution:** By hypothesis, there exists \( M > 0 \) such that \( |a_n| \leq M \) for all \( n \geq 1 \). Since \( b_n - b_{n+1} \geq 0 \) we have that \( |a_n(b_n - b_{n+1})| \leq M(b_n - b_{n+1}) \). By the linearity theorem \( \sum M(b_n - b_{n+1}) \) converges. By the comparison theorem, \( \sum |a_n(b_n - b_{n+1})| \) converges. Thus \( \sum_{n=1}^{\infty} a_n(b_n - b_{n+1}) \) converges.
6. Prove the following two statements:

(a) Every real number is a cluster point of some sequence of rational numbers.

**Solution:** Let \( r \) be a real number. For any \( n \geq 1 \), Theorem 2.5 on page 25 implies that there is a rational number \( a_n \) such that \( r < a_n < r + \frac{1}{n} \). By the squeeze theorem, \( \lim_{n \to \infty} a_n = r \). Theorem 6.2 implies that \( r \) is a cluster point for the sequence \( a_n \).

(b) Every real number is a cluster point of some sequence of irrational numbers.

**Solution:** The proof is similar with the previous part. If \( r \) is any real number, Theorem 2.5 implies that there is an irrational number \( a_n \) with \( r < a_n < r + \frac{1}{n} \) for all \( n \geq 1 \). Thus \( r = \lim_{n \to \infty} a_n \) and \( r \) is a cluster point of a sequence of irrational numbers.
7. Give examples for the following or explain why no example exists.

(a) A series that has bounded partial sums but does not converge.

**Solution:** Consider the sequence \( a_n = (-1)^n \) for all \( n \geq 1 \). Then \( s_{2k+1} = -1 \) and \( s_{2k} = 0 \) for all \( k \geq 1 \). Thus \( \{ s_n \} \) is a bounded sequence which diverges.

(b) A sequence which has an infinite number of cluster points.

**Solution:** Example 6.2A a) in the textbook says that for the sequence 1; 1, 2; 1, 2, 3; \ldots every integer is a cluster point.

(c) A power series whose radius of convergence is 2.

**Solution:** Consider the power series \( \sum \frac{x^n}{2^n} \). We compute the radius of convergence using the ratio test:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{2}.
\]

Thus the series is absolutely convergent if \( |x| < 2 \) and divergent for \( |x| > 2 \). Thus \( R = 2 \).
8. Prove, using the definition, that if \( \{a_n\} \) and \( \{b_n\} \) are Cauchy sequences then \( \{a_n b_n\} \) is Cauchy.

Solution: We proved in class that a Cauchy sequence is bounded (see also part (A) of the proof of Theorem 6.4). Thus there are \( M_1 > 0 \) and \( M_2 > 0 \) such that \( |a_n| \leq M_1 \) and \( |b_n| \leq M_2 \) for all \( n \geq 1 \). Let \( M = \max\{M_1, M_2\} \). Thus \( |a_n| \leq M \) and \( |b_n| \leq M \) for all \( n \geq 1 \). Let \( \varepsilon > 0 \). Since the sequence \( a_n \) is Cauchy there exists \( N_1 \geq 1 \) such that \( |a_n - a_m| < \frac{\varepsilon}{2M} \) for all \( m, n \geq N_1 \). Similarly there exists \( N_2 \geq 1 \) such that \( |b_n - b_m| < \frac{\varepsilon}{2M} \) for all \( m, n \geq N_2 \). Let \( N = \max\{N_1, N_2\} \). If \( n, m \geq N \) we have

\[
|a_n b_n - a_m b_m| = |a_n b_n - a_n b_m + a_n b_m - a_m b_m| \\
\leq |a_n b_n - a_n b_m| + |a_n b_m - a_m b_m| \\
= |a_n| |b_n - b_m| + |b_m| |a_n - a_m| \\
< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, \( \{a_n b_n\} \) is a Cauchy sequence.