

Name : \_\_\_\_\_

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**Instructions:**

1. Please write your name in the blank above, and sign & date below.
2. Please use the space provided to write your solution.
3. If you need extra pages please attach them at the end and clearly indicate each problem number where your solution begins.
4. Please staple your additional pages together *in order* with these pages on top.
5. This prelim is due by Wednesday, March 5, at 5:00pm. You may submit it during Wednesday morning's class or Wednesday afternoon in my office.

TIP: The exam will be significantly easier (and shorter) if you cite results (e.g., theorems and examples covered in lecture), rather than proving them from scratch. If you are unsure whether or not you are allowed to use certain things, please contact me. This exam contains 8 questions, for a total of 100 points. Good luck!

You may use your notes, results proved in the textbook, and results that you have proved in homework assignments (provided they are correct). You may use work done with others on past assignments, and work done in office hours with the instructor or the TA. Beyond this however, exam solutions must be entirely your own work. Upon receipt of this exam, you are on your honour not to discuss any material covered on this exam with anyone other than the instructor. Please sign below to indicate that you understand and agree:

Signature: \_\_\_\_\_ Date: \_\_\_\_\_

1	/20
2	/5
3	/5
4	/15
5	/10
6	/20
7	/15
8	/10
T	/100

1. Give an example of each of the following, or **use theorems** to prove that such a thing is impossible.

- (4pts) (a) Sequences  $\{a_n\}, \{b_n\}$  which both diverge, but whose sum  $\{a_n + b_n\}$  converges.

**Solution:** If  $a_n = (-1)^n$  and  $b_n = (-1)^{n+1}$ , then both sequences are divergent (we proved this in class). Their sum, however, equals 0 for all  $n$ . Thus  $a_n + b_n$  is convergent to 0.

- (4pts) (b) Sequences  $\{a_n\}, \{b_n\}$  where  $a_n \rightarrow L$ ,  $\{b_n\}$  diverges, but whose sum  $\{a_n + b_n\}$  converges.

**Solution:** This statement is impossible. We prove this by contradiction. Assume that  $\{a_n + b_n\}$  is convergent. Then, since the sequence  $\{a_n\}$  is also convergent, the linearity theorem implies that the sequence  $\{a_n + b_n - a_n\} = \{b_n\}$  converges. This is a contradiction.

- (4pts) (c) A convergent sequence  $\{a_n\}$  where  $a_n \neq 0$  for any  $n$ ,  $\{a_n\}$  converges, but  $\{\frac{1}{a_n}\}$  diverges.

**Solution:** If  $a_n = \frac{1}{n}$  for all  $n \geq 1$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ ,  $a_n \neq 0$  for all  $n \geq 1$ . Also, the limit  $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \lim_{n \rightarrow \infty} n = \infty$ .

- (4pts) (d) An unbounded sequence  $\{a_n\}$  and a convergent sequence  $\{b_n\}$  such that  $\{a_n - b_n\}$  is bounded.

**Solution:** This statement is impossible. We prove by contradiction. Suppose that there are  $M$  and  $L$  such that  $L \leq a_n - b_n \leq M$  for all  $n$ . We also know that there are two numbers  $A$  and  $B$  such that  $A \leq b_n \leq B$  for all  $n$ . Since  $a_n = b_n + (a_n - b_n)$  it follows that  $A + L \leq a_n \leq B + M$  for all  $n$ . This implies that  $\{a_n\}$  is a bounded sequence, which is a contradiction.

- (4pts) (e) Sequences  $\{a_n\}, \{b_n\}$  where  $\{a_n\}$  converges,  $\{a_n b_n\}$  converges, but  $\{b_n\}$  does not converge.

**Solution:** Let  $a_n = 1/n$ ,  $b_n = n$ . Then  $a_n$  converges (it's limit is 0),  $b_n$  diverges. Their product, however, is  $a_n b_n = 1$  for all  $n$ . This is a convergent sequence.

(Note: just to clarify, if  $a_n \rightarrow \infty$ , then  $\{a_n\}$  is *not* convergent.)

- (5pts) 2. If  $|a_n - L| < \frac{|L|}{2}$ , prove that  $|a_n| > \frac{|L|}{2}$ . (A picture will not suffice)

**Solution:** Using the triangle inequality we have:

$$|L| = |L - a_n + a_n| \leq |L - a_n| + |a_n| < \frac{|L|}{2} + |a_n|.$$

The last inequality follows from the hypothesis. Then  $|L| - \frac{|L|}{2} < |a_n|$ . The conclusion follows.

(5pts) 3. Using only the definition of a limit (and without using the limit laws) prove that

$$\lim_{n \rightarrow \infty} \frac{3n^3 + 4n}{n^3 + n + 1} = 3.$$

**Solution:** Let  $\varepsilon > 0$ . We need to show that there is  $N \geq 0$  such that if  $n \geq N$

$$\left| \frac{3n^3 + 4n}{n^3 + n + 1} - 3 \right| < \varepsilon$$

We solve this inequality by simplifying first the left-hand side:

$$\left| \frac{3n^3 + 4n}{n^3 + n + 1} - 3 \right| = \left| \frac{n - 3}{n^3 + n + 1} \right|$$

If  $n \geq 3$  then we can drop the absolute value bars and we can simplify in the following way:

$$\frac{n - 3}{n^3 + n + 1} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

It suffices to solve the inequality  $1/n^2 < \varepsilon$ . If we let  $N = \sqrt{1/\varepsilon} + 1$  then

$$\left| \frac{3n^3 + 4n}{n^3 + n + 1} - 3 \right| < \varepsilon$$

for all  $n \geq N$ . Since  $\varepsilon$  was arbitrarily chosen, it follows that

$$\lim_{n \rightarrow \infty} \frac{3n^3 + 4n}{n^3 + n + 1} = 3.$$

(15pts) 4. Suppose  $\{a_n\}$  converges to  $L$ . Define a new sequence  $\{b_n\}$  by

$$b_n = \frac{a_n + a_{n+1}}{2}$$

for all  $n$ . Prove that  $\{b_n\}$  converges to  $L$ .

**Solution:** Since  $\{a_{n+1}\}$  is a subsequence of  $\{a_n\}$  (because we pick all indices with the exception of the first one),  $\{a_{n+1}\}$  is convergent and  $\lim_{n \rightarrow \infty} a_{n+1} = L$  by the subsequence theorem. By linearity

$$\lim_{n \rightarrow \infty} b_n = \frac{\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} a_{n+1}}{2} = \frac{L + L}{2} = L.$$

(10pts) 5. Prove that, if  $\{a_n\}$  converges to  $L$  then  $\{|a_n|\}$  converges to  $|L|$ . Is the converse true?

**Solution:** We claim the following inequality holds for any two real numbers  $a$  and  $b$ :

$$||a| - |b|| \leq |a - b|.$$

Using the difference form of the triangle inequality we have that  $|a| - |b| \leq |a - b|$ . This is, however, only half of what we need. For the other half we can proceed as follows:

$$|b| = |b - a + a| \leq |b - a| + |a|.$$

Then, since  $|b - a| = |a - b|$ ,  $-|a - b| \leq |a| - |b|$ . Combining the two inequalities we obtain the claimed inequality.

Now let  $\varepsilon > 0$ . Since  $\{a_n\}$  converges to  $L$  there is  $N \geq 1$  such that  $|a_n - L| < \varepsilon$  for  $n \geq N$ . Thus

$$||a_n| - |L|| \leq |a_n - L| < \varepsilon$$

for  $n \geq N$ . Since  $\varepsilon$  was arbitrary  $\{|a_n|\}$  converges to  $|L|$ .

Finally, the converse is false. For example  $a_n = (-1)^n$  is divergent while  $|a_n| = 1$  converges to 1.

(20pts) 6. Prove that

$$\lim_{n \rightarrow \infty} \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} \right) = \frac{\pi}{4}.$$

**Solution:** Recall from page 52 of the textbook that

$$1 + a + a^2 + \cdots + a^n = \frac{1}{1-a} - \frac{a^{n+1}}{1-a}, \quad a \neq 1.$$

Substituting  $a = -u^2$  in the previous formula we obtain

$$1 - u^2 + u^4 - \cdots + (-1)^n u^{2n} = \frac{1}{1+u^2} - (-1)^{n+1} \frac{u^{2n+2}}{1+u^2}.$$

Integrating from 0 to 1 we obtain

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} = \frac{\pi}{4} - (-1)^{n+1} \int_0^1 \frac{u^{2n+2}}{1+u^2} du.$$

Using error term analysis, it follows that our sequence converges to  $\pi/4$  iff  $|e_n| = \int_0^1 \frac{u^{2n+2}}{1+u^2} du$  converges to 0. The proof is similar with the one did in class:

$$|e_n| \leq \int_0^1 u^{2n+2} du = \frac{1}{2n+3}.$$

Since the last sequence converges to 0 so does  $|e_n|$  and the proof is over.

7. Define a sequence recursively by  $a_{n+1} = \sqrt{2a_n}$ ,  $a_0 > 0$ .

- (10pts) (a) Prove that for any choice of  $a_0 > 0$ , the sequence  $a_n$  is monotone and bounded.
- (5pts) (b) Does the sequence has a limit? If yes, determine it's limit  $L$  and prove that it is the limit. What do you notice?

**Solution:**

- (a) We notice that the behavior of the sequence depends on the value of  $a_0$ . If  $a_0 = 2$  then the sequence is constant 2. Hence it is bounded and monotone.

If  $a_0 < 2$  then we prove first that  $a_n < 2$  for all  $n \geq 0$ . We accomplish this by induction. We have that  $a_0 < 2$ . Suppose now that  $a_n < 2$ . Then  $a_{n+1} = \sqrt{2a_n} < \sqrt{2 \cdot 2} = 2$ . Thus  $a_n < 2$  for all  $n \geq 0$ . We show next that the sequence is increasing. We do this by induction. We have that  $a_1 = \sqrt{2a_0} > \sqrt{a_0^2} = a_0$ . Assume now that  $a_n > a_{n-1}$ . Then  $a_{n+1} = \sqrt{2a_n} > \sqrt{2a_{n-1}} = a_n$ . Hence  $\{a_n\}$  is an increasing sequence bounded above by 2. Since  $\{a_n\}$  is also bounded below by  $a_0$ , the sequence is bounded.

If  $a_0 > 2$  then we can repeat the proof above by inverting the inequalities to obtain that  $\{a_n\}$  is a decreasing sequence bounded below by 2 and above by  $a_0$ .

- (b) Since the sequence is always bounded and monotone it will always have a limit by the completeness theorem. Let  $L = \lim_{n \rightarrow \infty} a_n$ . We proved in class that  $L = \lim_{n \rightarrow \infty} a_{n+1}$ . Also  $\lim \sqrt{2a_n} = 2L$ . Thus taking the limit in the equation  $a_{n+1} = \sqrt{2a_n}$  we obtain that  $L = \sqrt{2L}$ . Hence  $L = 2$  ( $L = 0$  is not possible by the properties we proved about our sequence). We notice that the limit is independent of the initial value  $a_0$ .

- (5pts) 8. (a) Let  $a_n = \sqrt{n^2 + n} - n$ . Find  $\lim_{n \rightarrow \infty} a_n$ . You can use limit theorems but carefully cite the theorems you use and explain why can you use them.

**Solution:** We have that

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n = \lim_{n \rightarrow \infty} \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{n(\sqrt{1 + 1/n} + 1)} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n} + 1} \\ &= \frac{1}{\lim_{n \rightarrow \infty} \sqrt{1 + 1/n} + 1} = \frac{1}{2},\end{aligned}$$

where the last two inequalities follow from the quotient and linearity theorem.

- (5pts) (b) Show that  $\lim_{n \rightarrow \infty} \sqrt{n^2 + 1} - n = 0$  using the definition of a limit.

**Solution:** For  $\varepsilon > 0$

$$\begin{aligned}|\sqrt{n^2 + 1} - n| &= \left| \frac{1}{\sqrt{n^2 + 1} + n} \right| \leq \frac{1}{\sqrt{n^2 + 1}} \\ &= \frac{1}{2n} < \frac{1}{n}.\end{aligned}$$

Thus if  $N = 1/\varepsilon$ ,  $|\sqrt{n^2 + 1} - n| < \varepsilon$  for  $n \geq N$ . Hence  $\lim_{n \rightarrow \infty} \sqrt{n^2 + 1} - n = 0$ .