Definition 1. A **field** is a set \( \mathbb{F} \) together with two binary operations\(^1\) \( + : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \) (addition) and \( \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \) (multiplication) and two distinct elements \( 0_\mathbb{F}, 1_\mathbb{F} \in \mathbb{F} \) (sometimes denoted simply by \( 0, 1 \), or even \( 0, 1 \)) that satisfy

(A1) For every \( a, b \in \mathbb{F} \), \( a + b = b + a \).

(A2) For every \( a, b, c \in \mathbb{F} \), \( (a + b) + c = a + (b + c) \).

(A3) For every \( a \in \mathbb{F} \), \( 0_\mathbb{F} + a = a \).

(A4) For each \( a \in \mathbb{F} \), there exists an element \( b \in \mathbb{F} \) satisfying \( a + b = 0_\mathbb{F} \). We often denote this element \( b \) by \( (-a) \).

(M1) For each \( a, b \in \mathbb{F} \), \( a \cdot b = b \cdot a \).

(M2) For each \( a, b, c \in \mathbb{F} \), \( a \cdot (b \cdot c) = (a \cdot b) \cdot c \).

(M3) For every \( a \in \mathbb{F} \), \( 1_\mathbb{F} \cdot a = a \).

(M4) For each \( a \neq 0_\mathbb{F} \) in \( \mathbb{F} \), there is an element \( b \in \mathbb{F} \) such that \( a \cdot b = 1_\mathbb{F} \). We often denote this element \( b \) by \( (1/a) \).

(M5) For each \( a, b, c \in \mathbb{F} \), \( a \cdot (b + c) = a \cdot b + a \cdot c \).

Definition 2. A **vector space** over a field \( \mathbb{F} \) is a set \( V \) together with two binary operations \( + : V \times V \to V \) (vector addition) and \( \cdot : \mathbb{F} \times V \to V \) (scalar multiplication), and an element \( \vec{0}_V \) (sometimes simply denoted as \( 0 \), or \( 0 \)), that satisfy

(A1) For every \( u, v \in V \), \( u + v = v + u \).

(A2) For every \( u, v, w \in V \), \( (u + v) + w = u + (v + w) \).

(A3) For every \( v \in V \), \( \vec{0}_V + v = v \).

(A4) For each \( v \in V \), there exists an element \( w \in V \) satisfying \( v + w = \vec{0}_V \). We often denote this element \( w \) by \( (−v) \).

(M1) For each \( a \in \mathbb{F} \) and \( u, v \in V \), \( a \cdot (u + v) = a \cdot u + a \cdot v \).

(M2) For each \( a, b \in \mathbb{F} \) and \( v \in V \), \( (a + b) \cdot v = a \cdot v + b \cdot v \).

(M3) For each \( a, b \in \mathbb{F} \) and \( v \in V \), \( (a \cdot b) \cdot v = a \cdot (b \cdot v) \).

(M4) For each \( v \in V \), \( 1_\mathbb{F} \cdot v = v \).

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\(^1\)Saying that \( + \) and \( \cdot \) are binary operations implicitly assumes that they are well-defined.
Definition 3. A ring (with identity) is a set $R$ together with two operations $+: R \times R \to R$ (addition) and $\cdot : R \times R \to R$ (multiplication), and two elements $0_R$ and $1_R$ (sometimes denoted simply by 0 and 1), that satisfy

(A1) For every $a, b \in R$, $a + b = b + a$.

(A2) For every $a, b, c \in R$, $(a + b) + c = a + (b + c)$.

(A3) For every $a \in R$, $0_R + a = a$.

(A4) For each $a \in R$, there exists an element $b \in R$ satisfying $a + b = 0_R$. We often denote this element $b$ by $(-a)$.

(M2) For each $a, b, c \in R$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(M3) For each $a \in R$, $1_R \cdot b = b \cdot 1_R = b$.

(M5) For each $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$, and $(a + b) \cdot c = a \cdot c + b \cdot c$.

If also

(M1) For each $a, b \in R$, $a \cdot b = b \cdot a$,

we call $R$ a **commutative ring (with identity)**. In this class all rings have an identity.

Warning: The numbering $M1, M2, etc$ for rings does not match the numbering for vector spaces!

Definition 4. An algebra (with identity) over the field $\mathbb{F}$ is a vector space $A$ over $\mathbb{F}$, together with an extra operation $\mu : A \times A \to A$ (multiplication, often simply written as $\mu(a, b) = ab$), and an element, $1_A$, that, in addition to the vector space axioms, satisfy

(M1) For each $a, b, c \in A$, $a(bc) = (ab)c$.

(M2) For each $a, b, c \in A$, $a(b + c) = ab + ac$, and $(a + b)c = ac + bc$.

(M3) For each $a, b, c \in A$, and $\gamma \in \mathbb{F}$, $(\gamma \cdot a)b = \gamma \cdot (ab) = a(\gamma \cdot b)$.

(M4) For every $a \in A$, $1_A b = b 1_A = b$.

If also

(M5) For each $a, b \in A$, $ab = ba$,

we call $A$ a **commutative algebra (with identity)**. In this class all algebras have an identity.
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