1. (a) Equivalent:
   -Let $T$ be such that $\ker T = 0$. From class, $T$ injective. By rank-nullity, $\dim(\im(T)) = \dim(V) = n$, so $n$ finite $\implies \im(T) = V$. Hence $T$ is also surjective so $T$ invertible.
   -If $T$ invertible, then $T(a) = 0$ for only one $a$ (namely $a = 0$), and thus $\ker T = 0$.

(b) Equivalent:
   -If $\im(T) = V$ then by rank-nullity $\dim(\ker T) = 0 \implies \ker T = 0$.
   -If $\ker T = 0$, rank-nullity $\implies \dim(\im(T)) = \dim(V) = n$, so $n$ finite $\implies \im(T) = V$.
   -Thus $\im(T) = V \iff \ker T = 0 \iff T$ invertible.

(c) Equivalent:
   -$\rank(T) = n \iff \dim(\im(T)) = \dim(V) = n \iff \im(T) = V \iff T$ invertible

(d) Not Equivalent:
   -Let $V = \mathbb{R}$ and let $T$ be the zero map (so $T(x) = 0$ for all $x \in \mathbb{R}$, note that $T \in L(\mathbb{R}, \mathbb{R})$).
   Then $\ker T = \mathbb{R} = V$ but clearly $T$ is not invertible.

(e) Not Equivalent:
   -Consider $A = [T]_{A \leftarrow A}$ the zero matrix. Then $rref(A) = 0$ as well, which has $n$ free columns. But clearly $A$ corresponds to the zero map $T$, which is not invertible.

(f) Equivalent:
   -From 1a last week, (# pivots of $rref(A)) = \rank(A)$. Thus equivalence follows by (c).

(g) Equivalent:
   -This equivalence is exactly Theorem 2.33 (pg. 83) in the text.

(h) Equivalent:
   -Let $D$ denote the change of basis map, taking $[v]_A$ to $[v]_B$. So $C = DA \iff D^{-1}C = A$.
   -Then if $A$ invertible by $A^{-1}$, $C$ invertible by $A^{-1}D^{-1}$ ($CA^{-1}D^{-1} = DAA^{-1}D^{-1} = I$).
   -And if $C$ invertible by $C^{-1}$, then $A$ invertible by $C^{-1}D$ ($AC^{-1}D = D^{-1}CC^{-1}D = I$).
   -Thus $C$ invertible $\iff A$ invertible $\iff T$ invertible

(i) Not Equivalent:
   -Let $V = \mathbb{R}$ and let $T$ be the zero map. Then there is an isomorphism $V/\ker(T) \to \im(T)$ by the first isomorphism theorem, but clearly $T$ is not invertible.

2. (a) -Let $f, f' \in \langle f_1(x), \ldots, f_r(x) \rangle = I$, $a \in \mathbb{F}$, and $g \in \mathbb{F}[x]$.
   -Then $f = g_1 f_1 + \cdots + g_r f_r$ and $f' = g'_1 f_1 + \cdots + g'_r f_r$ for $g_i, g'_i \in \mathbb{F}[x]$. So $f + f' = g_1 f_1 + \cdots + g_r f_r + g'_1 f_1 + \cdots + g'_r f_r = (g_1 + g'_1) f_1 + \cdots + (g_r + g'_r) f_r \in I$ for $(g_i + g'_i) \in \mathbb{F}[x]$.
   -And $af = (ag_1 f_1 + \cdots + ag_r f_r) \in I$ for $ag_i \in \mathbb{F}[x]$. Clearly $I$ nonempty, so $I$ is a subspace.
   -Further, $fg = (g_1 f_1 + \cdots + g_r f_r)g = ((gg_1) f_1 + \cdots + (gg_r) f_r) \in I$ for $gg_i \in \mathbb{F}[x]$.
   -Since $\mathbb{F}[x]$ commutative, $gf = fg \in I$ and so $I$ is an ideal by definition.
(b) - Consider an ideal $I \neq \{0\}$ in $Mat_{n\times n}(F) = M$. Then $I$ contains a nonzero matrix $A$.
- Consider $B \in M$ arbitrary. We will show that $B \in I$, thus $I = M$ and so $M$ simple:
  - Let $B_{ij}$ be the matrix that is 0 everywhere but the $(i,j)$th position, which contains $b_{ij}$.
  - Let a nonzero entry in $A$ be $a_{hk}$. Let $C$ be a matrix that is 0 everywhere except $c_{hk} = 1$.
  - Then $A' = CAC \in I$, where $A'$ is a matrix with 0 everywhere but $a'_{hk} = a_{hk}$.
  - Now, using elementary row and column swaps on $A'$ (which again result in an element of $I$), we move $a_{hk} = a'_{hk}$ to position $(i,j)$. Now multiply by $a^{-1}_{hk}b_{ij}$, to obtain $B_{ij} \in I$.
- Thus for any $(i,j)$ we have that $B_{ij} \in I$. Since $I$ is an ideal, we can sum over all $(i,j)$ to obtain $B \in I$.

3. (a) - Let $f, f' \in \text{ann}(A) = \{f(x) \in F[x] \mid f(A) = 0\}$, $c \in F$, and $g \in F[x]$.
  - Then $f(A) + f'(A) = 0 \implies (f + f') \in \text{ann}(A)$, and $cf(A) = c0 = 0 \implies cf \in \text{ann}(A)$.
  - Further $fg = gf \in \text{ann}(A)$ because $f(A)g(A) = 0$, so $\text{ann}(A)$ is an ideal as hoped.

(b) - First, notice that $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (so $a_{21} = 1$). Therefore, $f(a_0 + a_1x + \cdots + a_nx^n) \in F[x]$ such that $a_0 = a_1 = 0$ will belong to $\text{ann}(A)$ (as $f(A) = a_2A^2 + \cdots + a_nA^n = 0$).
  - Now consider arbitrary $(f = a_0 + a_1x + \cdots + a_nx^n) \in \text{ann}(A)$. Since $f' = a_2x^2 + \cdots + a_nx^n \in \text{ann}(A)$ we have $f - f' \in \text{ann}(A)$ because $\text{ann}(A)$ an ideal.
  - Thus $(f - f')(A) = a_0I + a_1A = 0 \iff \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence $a_0 = a_1 = 0$.
  - So in fact $(f = a_0 + a_1x + \cdots + a_nx^n) \in \text{ann}(A) \iff a_0 = a_1 = 0$. Thus we have that $\text{ann}(A) = \{f = a_0 + a_1x + \cdots + a_nx^n \in F[x] \mid a_0 = a_1 = 0\}$.

4. (a) Assume $f$ has inverses $g$ and $g'$. Then $g' = (gf)g' = g(fg') = g$, so $g = g' = f^{-1}$ unique.
(b) Since $	ext{dim}(V) = n, V \simeq F^n$ for any $F^n$ of dimension $n$. The isomorphism gives a 1-to-1 correspondence between bases of $V$ and bases of $F^n$. Similarly, there is a 1-to-1 correspondence between $L(V, V)$ and $L(F^n, F^n)$. So it’s enough to show the claim for $F^n$:
- Consider function $f : \{T \in L(F^n, F^n) \mid T \text{ invertible}\} \rightarrow \{\text{ordered bases of } F^n\}$ defined by $f(T) = \{T_e_1, \ldots, T_e_n\}$ (where $S = \{e_1, \ldots, e_n\}$, noting by 1c that $T$ invertible implies $f(T)$ a basis for $F^n$)
  - Then if $f(T) = f(T')$, we have $[T]_{S \leftarrow S} = [T']_{S \leftarrow S}$ and so $T = T'$. Thus $f$ injective.
  - Further, let $([v_1]_S, \ldots, [v_n]_S) \in F^n$ an ordered basis. Consider the transformation $T$ that maps $e_i$ to $[v_i]_S$. The matrix $[T]_{S \leftarrow S}$ corresponding to $T$ is the change of basis matrix, with columns $[v_1]_S (T$ invertible by 1c). Then $f(T) = ([v_1]_S, \ldots, [v_n]_S)$ so $f$ surjective.
  - Thus $f$ a bijection and we have a one-to-one correspondence between units in $L(F^n, F^n)$ and ordered bases of $F^n$.

(c) From homework 5 number 5, we saw that $F^3_3$ has $(3^3 - 1)(3^3 - 3)(3^3 - 3^2) = c$ ordered bases. So from part (b), $c$ is also the number of units in $L(F^3_3, F^3_3)$.

5. - Let $S : V \rightarrow V$ nilpotent, so $S^k$ the zero map for a $k \in \mathbb{N}$. Then $(I - S)(I + S + S^2 + \cdots + S^{k-1}) = (I + S + S^2 + \cdots + S^{k-1}) - (S + S^2 + S^3 + \cdots + S^k) = I - S^k = I$, so $(I - S)$ invertible.

6. - Let $A, B, C \in Mat_{n\times n}(F)$. Then $A \sim A$ because $A = I^{-1}AI$.
  - Further, if $A \sim B$ then $A = P^{-1}BP \implies PAP^{-1} = B$ so $B \sim A$.
  - Lastly, if $A \sim B$ and $B \sim C$ then $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Hence $A = P^{-1}Q^{-1}CQP = (QP)^{-1}C(QP)$ and $A \sim C$. Thus $\sim$ is an equivalence relation.