Let $T : V \rightarrow W$ and $S : W \rightarrow U$ be linear transformations.

(a) Let $v_1, v_2 \in V$ and $\alpha \in \mathbb{F}$. Then by the linearity of $S$ and $T$, we have $S(T(\alpha v_1)) = S(\alpha T(v_1)) = \alpha S(T(v_1))$ and $S(T(v_1 + v_2)) = S(T(v_1)) + S(T(v_2))$. Hence $S \circ T$ is linear by definition.

(b) Assume $S : W \rightarrow U$ is also an isomorphism, so we can consider $S^{-1} : U \rightarrow W$. Let $u_1, u_2 \in U$ and $\alpha \in \mathbb{F}$. $S$ is surjective, thus $u_i = S(w_i) \iff S^{-1}(u_i) = w_i$. Hence $S \circ T$ is also an isomorphism, so we can consider $S^{-1}(w_1) = \alpha (S^{-1}(u_1)) = \alpha w_1 = \alpha S^{-1}(u_1)$ and $S^{-1}(u_1 + u_2) = \alpha S^{-1}(u_1) = S^{-1}(S(w_1 + w_2)) = S^{-1}(S(w_1 + w_2)) = w_1 + w_2 = S^{-1}(u_1) + S^{-1}(u_2)$. Hence $S^{-1}$ is linear by definition.

(c) Assume $S$ and $T$ surjective and let $u \in U$. Then $S$ surjective implies $u = S(w)$ and $T$ surjective implies $w = T(v)$. Thus we have $u = S(T(v))$, showing $S \circ T$ surjective.

(d) Assume $S$ and $T$ injective with $S(T(v_1)) = S(T(v_2))$. Since $S$ is injective, $T(v_1) = T(v_2)$. Then $T$ injective implies $v_1 = v_2$, showing $S \circ T$ injective.

2. Let $V = X \oplus Y$

(a) For all $v \in V$ we have $P_1(v) = P_1(x + y) = x$. Thus $P_1(P_1(v)) = P_1(x + 0) = x = P_1(v)$, showing $P_1 \circ P_1 = P_1$.

(b) For all $v \in V$ we have $P_1(v) + P_2(v) = P_1(x + y) + P_2(x + y) = x + y = 1_V(x + y) = 1_V(v)$, showing $P_1 + P_2 = 1_V$.

(c) For all $v \in V$ we have $P_1(P_2(v)) = P_1(P_2(x + y)) = P_1(y) = 0$, showing $P_1 \circ P_2 = 0_{V \rightarrow V}$.

3. Let $T : V \rightarrow V$ a linear transformation

(a) Assume $T^2 = T$.

Lemma: If $im(T) \cap \ker(T) = 0$ then we have $V = \ker(T) \oplus im(T)$

Proof: Let $v \in V$. It is enough to show that $v - T(v) \in \ker(T)$, so $v - T(v) = w \iff v = w + T(v)$ is a decomposition showing $V = \ker(T) + im(T)$. Since $T^2 = T$ is linear, $T(v - T(v)) = T(v) - T(T(v)) = 0$ proves the claim.
By the lemma, it is enough to show that and \( \text{im} T \cap \ker(T) = 0 \) and \( \text{im}(T) = \ker(T - 1_V) \), then \( V = \ker(T) \oplus \text{im}(T) \implies V = \ker(T) \oplus \ker(T - 1_V) \).

First we show that \( \ker(T) \cap \text{im}(T) = 0 \). Assume \( v \in \ker(T) \cap \text{im}(T) \). So \( T(v) = 0 \) and \( T(w) = v \implies T(v) = T(T(w)) = 0 \). But since \( T^2 = T \), \( T(T(w)) = T(w) = v \), thus we have \( v = 0 \).

Now let \( v \in \ker(T - 1_V) \), so \( T(v) - v = 0 \implies T(v) = v \). Then by definition \( v \in \text{im}(T) \), so we have \( \ker(T - 1_V) \subseteq \text{im}(T) \).

Now let \( v \in \text{im}(T) \), then \( v = T(w) \) and so \( (T - 1_V)(v) = (T - 1_V)(T(w)) = T(T(w)) - T(w) = T(w) - T(w) = 0 \). Thus \( v \in \ker(T - 1_V) \) and so \( \text{im}(T) \subseteq \ker(T - 1_V) \). Thus we see \( \ker(T - 1_V) = \text{im}(T) \).

Example: When \( V = \mathbb{R}^2 \), consider \( T(x, y) = (x, 0) \). Then \( T(a(x, y)) = T(ax, ay) = (ax, 0) = a(T(x, y) \text{ and } T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2) = (x_1 + x_2, 0) = T(x_1, y_1) + T(x_2, y_2) \) so \( T \) is linear. And we have for any \( (x, y) \in \mathbb{R}^2 \) that \( T(T(x, y)) = T(x, 0) = (x, 0) = T(x, y) \).

(b) Assume \( V = \ker(T) \oplus \ker(T - 1_V) \).

Then for any \( v \in V \), there are \( w \in \ker(T) \) and \( u \in \ker(T - 1_V) \) so \( v = w + u \). Then \( T(v) = T(w + u) = T(w) + T(u) = T(u) \). Further, \( T(T(v)) = T(T(w + u)) = T(T(w)) + T(T(u)) = T(0) + T(T(u)) = T(T(u)) \).

Since \( u \in \ker(T - 1_V) \), \( T(u) - u = 0 \implies T(T(u) - u) = T(T(u)) - T(u) = 0 \). Hence \( T(v) = T(u) = T(T(u)) = T(T(v)) \) for all \( v \) and so \( T = T^2 \).

(c) When \( V = \mathbb{R}^2 \), consider \( T(x, y) = (-y, x) \). Then \( T(a(x, y)) = T(ax, ay) = (-ay, ax) = a(-y, x) = aT(x, y) \) and \( T((x_1, y_1) + (x_2, y_2)) = T(x_1 + x_2, y_1 + y_2) = (-y_1 + y_2, x_1 + x_2) = (-y_1, x_1) + (-y_2, x_2) = T(x_1, y_1) + T(x_2, y_2) \) so \( T \) is linear. And we have for any \( (x, y) \in \mathbb{R}^2 \) that \( T(T(x, y)) = T(-y, x) = (-x, -y) = -1_V(x, y) \).

(d) Let \( T^2 = 0_{V \rightarrow V} \). If \( \dim(V) = \infty \) then \( \text{rank}(T) \leq \frac{\dim(V)}{2} = \infty \).

Else let \( \dim(V) \) be finite. Assume \( \text{rank}(T) > \frac{\dim(V)}{2} \), so by rank nullity, \( \text{nullity}(T) < \frac{\dim(V)}{2} \). Then we can find a basis \( \{v_1 = T(w_1), \ldots, v_m = T(w_m)\} \) for the image of \( T \) where \( m > \frac{\dim(V)}{2} \) vectors. But then \( T(T(w_i)) = 0 \) for all \( i \in \{1, \ldots, m\} \) contradicts the dimension of \( \text{nullity}(T) \). Hence \( \text{rank}(T) \leq \frac{\dim(V)}{2} \).

4. Let \( V = U_1 \oplus W = U_2 \oplus W \)

(a) Lemma: If \( V = U \oplus W \) then \( V/W \cong U \) (and similarly \( V/U \cong W \))

Proof: Consider the map \( T : V \rightarrow U \) defined by \( T(v) = T(u + w) = u \).

Then \( T(av) = au = aT(v) \) and \( T(v_1 + v_2) = T(u_1 + w_1 + u_2 + w_2) = u_1 + u_2 = \)
\( T(v_1) + T(v_2) \). Thus \( T \) linear.

For any \( u \in U \), we have \( u \in V \) so \( T(u) = T(u + 0) = u \) shows that \( T \) is a surjection.

Let \( T(v) = T(u + w) = u = 0 \). Then we have that \( v = w \) so \( v \in W \). Thus \( \ker(T) = W \).

Thus by the first isomorphism theorem, \( V/W \cong U \).

Now \( V = U_1 \oplus W = U_2 \oplus W \implies (U_1 \oplus W)/W \cong (U_2 \oplus W)/W \cong U_1 \cong U_2 \).

Since we have shown in 1 that a composition of linear bijections is a linear bijection, a composition of isomorphisms is an isomorphisms and so \( U_1 \cong U_2 \).

(b) No, for example take \( V = \mathbb{R}^2 \), \( W = \text{span}((0,1)) \), \( U_1 = \text{span}((1,0)) \) and \( U_2 = \text{span}((1,1)) \). Then \( U_1 \cap W = U_2 \cap W = 0 \) and \( U_1 + W = U_2 + W = V \), but clearly \( U_1 \neq U_2 \).

5. Let \( f(x) \in C^\infty(\mathbb{R}) \) and consider \( e(x) = \frac{1}{2}(f(x) + f(-x)) \), \( o(x) = \frac{1}{2}(f(x) - f(-x)) \).

By definition, \( e(-x) = \frac{1}{2}(f(-x) + f(x)) = e(x) \implies e(x) \in E(\mathbb{R}) \) and \( o(-x) = \frac{1}{2}(f(-x) - f(x)) = -o(x) \implies o(x) \in O(\mathbb{R}) \). Since \( e(x) + o(x) = f(x) \) for all \( x \), we have that \( E(\mathbb{R}) + O(\mathbb{R}) = C^\infty(\mathbb{R}) \).

Further, \( f(x) \in O(\mathbb{R}) \cap E(\mathbb{R}) \implies f(-x) = -f(x) = f(x) \implies f(x) = 0 \) for all \( x \).

Thus \( E(\mathbb{R}) \oplus O(\mathbb{R}) = C^\infty(\mathbb{R}) \).

Now by the lemma from problem 4 we have \( C^\infty(\mathbb{R})/O(\mathbb{R}) \cong E(\mathbb{R}) \).