Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the “extended glossary” on separate paper (LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please staple this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.

Exercises.

1. Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$, and let $U, W$ be subspaces. Prove that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

2. Suppose that $V$ is a vector space over $\mathbb{F}$ of dimension 5. Suppose that $U$ and $W$ are subspaces, both of dimension 3. Show that the intersection $U \cap W$ is not zero. What are the possible dimensions for $U \cap W$? (of course, give reasons!)

3. Suppose that $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$ spans the vector space $V$ (over a field $\mathbb{F}$). Suppose that $\vec{v}_1$ is in the span of $(w, \vec{v}_2, \ldots, \vec{v}_n)$. Show that $(w, \vec{v}_2, \ldots, \vec{v}_n)$ spans $V$. (We used this in our proof of the exchange theorem).

4. Let $V = \mathbb{F}^3$, where $\mathbb{F} = \mathbb{F}_3$, and let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

   (a) Prove that $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis of $V$.

   (b) Compute the coordinate vectors for the standard basis of $\mathbb{F}^3$, with respect to the basis $\mathcal{B}$.
5. (a) If $\mathbb{F} = \mathbb{F}_p$, where $p$ is a prime number, and $V$ is a vector space of dimension $n$ over $\mathbb{F}$, how many elements does $V$ have?

(b) How many (ordered) bases do the vector spaces $\mathbb{F}^1$ and $\mathbb{F}^2$ have, if $\mathbb{F} = \mathbb{F}_2$? If $\mathbb{F} = \mathbb{F}_p$?

(c) How many (ordered) bases does the vector spaces $\mathbb{F}^3$ have, if $\mathbb{F} = \mathbb{F}_2$? If $\mathbb{F} = \mathbb{F}_p$?

6. Let $\mathbb{F}$ be a field with at least 3 elements, and let $a \in \mathbb{F}$ be an element which is not 0 or 1. Let $W \subset V = \mathbb{F}[x]$ be defined to be the set of polynomials which vanish at the points 0, 1, and $a$, i.e.

$$W = \{ f(x) \in \mathbb{F}[x] \mid f(0) = f(1) = f(a) = 0 \}.$$ 

(a) Is $W$ finite dimensional?

(b) Find 3 polynomials $f_0, f_1, f_a$ which take the value 1 at 0, 1, $a$, respectively, but vanish on the other two points in $\{0, 1, a\}$ (e.g. $f_0(0) = 1, f_0(1) = 0, f_0(a) = 0$).

(c) Find a complement $U$ to $W$ in $\mathbb{F}[x]$.

(d) Is $U$ finite dimensional? Find a basis for $U$.

(e) Is $\mathbb{F}[x]/W$ finite dimensional? Find a basis for this vector space.

Extended Glossary. There is no extended glossary this week.

Journal entry. There is no journal entry this week.

You may work in groups, but please write up your solutions yourself. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member’s extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.