1. (a) Assume \( \mathbf{a} \mathbf{u} = \mathbf{b} \mathbf{v} \). If \( a \neq 0 \) then \( a \) has multiplicative inverse \( a^{-1} \), so we must have \( a^{-1} \mathbf{a} \mathbf{u} = a^{-1} \mathbf{a} \mathbf{v} \implies \mathbf{u} = \mathbf{v} \).

(b) Assume \( \mathbf{a} \mathbf{u} = \mathbf{b} \mathbf{u} \). Then \( \mathbf{a} \mathbf{u} - \mathbf{b} \mathbf{u} = (a - b)\mathbf{u} = \mathbf{0} \). So from homework 3, we must have either \( a - b = 0 \) or \( \mathbf{u} = \mathbf{0} \). Hence, either \( a = b \) or \( \mathbf{u} = \mathbf{0} \).

(c) Let \( \mathbf{v} \in V \). Then \( (0_F + 0_F) \mathbf{v} = 0_F \mathbf{v} \) and so \( 0_F \mathbf{v} + 0_F \mathbf{v} = 0_F \mathbf{v} \). Since \( 0_F \mathbf{v} \) has an additive inverse, we may conclude \( 0_F \mathbf{v} = \mathbf{0} \).

(d) Let \( c \in F \) and consider \( c \mathbf{0} = \mathbf{v} \) for some \( \mathbf{v} \in V \). If \( c = 0_F \) then \( c \mathbf{0} = \mathbf{0} \) by part (c). Otherwise \( c \) has multiplicative inverse \( c^{-1} \) and so \( cc^{-1} \mathbf{0} = 0_F = c^{-1} \mathbf{v} \). Then since \( c^{-1} \) cannot be \( 0_F \), we conclude from homework 3 that \( \mathbf{v} = 0 \) in this case as well.

(e) Notice that \( cv + (-c)v = (c - c)v = 0_F v = 0 \) from part c. Then \( cv + (-c)v = cv - (cv) \), so using the additive inverse of \( cv \) we obtain \( (-c)v = -(cv) \).

2. For \( a, b, p \in \mathbb{Q} \), consider the equation \( \begin{pmatrix} 2 \\ a - b \\ 1 \end{pmatrix} = p \begin{pmatrix} a \\ b \\ 3 \end{pmatrix} \). This holds when \( 2 = pa \), \( a - b = pb \) and \( 1 = 3p \). From the third equation, take \( p = \frac{1}{3} \). Then from the first equation we take \( a = 6 \). And lastly we find that \( 6 - b = \frac{1}{3}b \implies b = 6 + \frac{3}{4} = \frac{9}{2} \). Thus, when \( a = 6, b = \frac{9}{2} \) these vectors are linearly dependent since

\[
\begin{pmatrix} 2 \\ \frac{3}{2} \\ 1 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 6 \\ \frac{9}{2} \\ 3 \end{pmatrix} = 0
\]

3. (a) Assume \( a \sin^2 x + b \cos^2 x = 0 \). Then when \( x = \frac{\pi}{2} \) we see that \( a1^2 + b0^2 = a = 0 \). And when \( x = 0 \) we see that \( a0^2 + b1^2 = b = 0 \). Hence, we must have \( a = b = 0 \), so these are linearly independent.

(b) Since \( 1 + (-1) \sin^2 x + (-1) \cos^2 x = 0 \), these are linearly dependent by definition.

(c) Assume \( ae^x + be^{2x} = 0 \). Since this must be satisfied at \( x = 0 \), we have that \( a + 0 + be^0 = a + b = 0 \). Similarly, considering \( x = 1 \) shows \( ae + be^2 = 0 \). Thus we
must have that \(-be + be^2 = b(e^2 - e) = 0\). Dividing by \(e^2 - e\) we obtain \(b = 0\) and so \(a + 0 = 0 \implies a = 0\). Since we must have \(a = b = 0\), these are linearly independent.

4. (a) Claim: \(\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}\) is a basis for \(V\) and \(\dim(V) = 4\)

Proof: First note that these are linearly independent since
\[
\begin{align*}
 a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\Rightarrow a &= b = c = d = 0
\end{align*}
\]

Now consider an arbitrary element of \(V\), which must have form \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) for some \(a, b, c, d \in \mathbb{F}\). We have that
\[
\begin{align*}
 a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{align*}
\]

Thus our set is a linearly independent spanning set and forms a basis for \(V\), so \(\dim(V) = 4\) by definition.

(b) i. Recall that a subset is a subspace iff it is closed under addition and scaling.
Consider \(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in V\).

Note that \(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \ast \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \ast \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\).

Thus, both of these matrices are in \(W\).
However, their sum is not in \(W\) because
\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ast \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 + 1 & 1 + 1 \\ 1 + 1 & 1 + 1 \end{pmatrix} \text{ (note that } 1 + 1 \neq 1 \text{ since } 1 \neq 0)\]

Hence, this is not a subspace.

ii. Let \(A, A' \in W\). Then \((A + A')B = AB + A'B = BA + BA' + B(A + A')\) so \((A + A') \in W\). Further, for any scalar \(c \in \mathbb{F}\), \((cA)B = cAB = cBA = B(cA)\) so \((cA) \in W\). Since \(W\) is closed under addition and scaling, it is a subspace.

Claim: \(\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}\) is a basis for \(W\) and \(\dim(W) = 2\)

Proof: These are elements of \(W\) because
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ast \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \ast \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\]

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and
\[
\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

Further, these are linearly independent since
\[
a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies a = b = 0
\]

Now consider an arbitrary element of \( W \) with form \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) for some \( a, b, c, d \in \mathbb{F} \). Since this matrix is in \( W \) we must have
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies \begin{pmatrix} a + b & b \\ c + d & d \end{pmatrix} = \begin{pmatrix} a + c & b + d \\ c + d & d \end{pmatrix}
\]

So we see that \( c + d = a + c \), and \( d = b + d \). Using the additive inverses of \( c \) and \( d \), we conclude that \( d = a \) and \( b = 0 \). Thus we can rewrite our element of \( W \) as \( \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \).

Thus our set is a linearly independent spanning set and forms a basis for \( W \), so \( \dim(W) = 2 \) by definition.

5. (a) For \( a = (a_0, a_1, \ldots) , b = (b_0, b_1, \ldots) \in \ell^\infty(\mathbb{F}) \) and \( c \in \mathbb{F} \) define \( a + b = (a_0 + b_0, a_1 + b_1, \ldots) \in \ell^\infty(\mathbb{F}) \) and \( ca = (ca_0, ca_1, \ldots) \in \ell^\infty(\mathbb{F}) \).

Then the zero element is \((0, 0, \ldots)\) since \( a + (0, 0, \ldots) = a \).

(b) Claim: \( W \) is finitely generated by the basis
\[
\left\{w_0 = (w^0_0 = 1, w^0_1 = 0, w^0_2, \ldots), w_1 = (w^1_0 = 0, w^1_1 = 1, w^1_2, \ldots) \mid w^i_j = w^i_{j-1} + w^i_{j-2} \text{ for } i \geq 2\right\}
\]

Proof: These are both elements of \( W \) by definition. They are linearly independent because
\[
a(1, 0, \ldots) + b(0, 1, \ldots) = (a, b, \ldots) = (0, 0, \ldots) \implies a = b = 0
\]

Lastly, let \((a_0, a_1, a_2, \ldots) \in W \) be arbitrary and consider \( a_0 w_0 + a_1 w_1 = (a_0, a_1, \ldots) \). Since these two vectors have the same first two coordinates, and they are both in the subspace \( W \), they must be equal.

Thus our set is a linearly independent spanning set and forms a basis for \( W \), so \( \dim(W) = 2 \) by definition.
6. (a) Assume $U = \text{span}\{(e_i \mid i \geq 2)\}$, then $U \oplus W = \ell^\infty(F)$

Proof: By the definition of $W$, a nonzero vector $v = (v_0, v_1, \ldots) \in W$ must have that either $v_0$ or $v_1$ is nonzero. However $u \in U = \text{span}\{(e_i \mid i \geq 2)\}$ must have the first two coordinates 0, so we have $U \cap W = 0$.

Thus, from class, we will be done if we can show $U + W = \ell^\infty(F)$. Any $u + w \in U + W$ is in $\ell^\infty(F)$ since both $u, w \in \ell^\infty(F)$. Thus $U + W \subseteq \ell^\infty(F)$. Now, let $a = (a_0, a_1, a_2, \ldots) \in \ell^\infty(F)$. Then by algebra we see $(a_0, a_1, a_2, \ldots) = (a_0w_0 + a_1w_1) + \sum_{i=2}^\infty (a_i - (a_0w_i^0 + a_1w_i^1))e_i$. Therefore $\ell^\infty(F) \subseteq U + W$ and so $\ell^\infty(F) = U + W$ as hoped.

Claim: $\{[w]_u, [w]_1\}$ is a basis for $\ell^\infty(F)/U$, so dim($\ell^\infty(F)/U) = 2$

Proof: Let $a[w]_0 + b[w]_1 = [aw]_0 + [bw]_1 = [0]_u$. Then we must have $aw_0 + bw_1 = (a, b, \ldots) = u \in U$ by definition. However, then $(a, b, \ldots) = (0, 0, \ldots) \implies a = b = 0$, so our set is linearly independent.

Consider an arbitrary element $[(a_0, a_1, a_2, \ldots)]_u \in \ell^\infty(F)/U$. Since $(0, 0, a_2, \ldots) \in U$ we have $[(a_0, a_1, a_2, \ldots)]_u = [(a_0, a_1, 0, \ldots)]_u + [(0, 0, a_2, \ldots)]_u = [(a_0, a_1, 0, \ldots)]_u + [0]_u = [(a_0, a_1, 0, \ldots)]_u = a_0[w]_0 + a_1[w]_1$.

Thus our set is a linearly independent spanning set and forms a basis, so $\text{dim}(\ell^\infty(F)/U) = 2$ by definition.

6. (a) Assume $U \cap W = 0$. Since $U, W$ are subspaces of finite dimensional $V$, they are also finite dimensional. Take a basis for $U$ to be $\{u_1, \ldots, u_n\}$, and a basis for $W$ to be $\{w_1, \ldots, w_m\}$.

Claim: $\{u_1, \ldots, u_n\} \cup \{w_1, \ldots, w_m\}$ is a basis for $U + W$

Proof: Assume $\alpha_1u_1 + \cdots + \alpha_nu_n + \beta_1w_1 + \cdots + \beta_mw_m = 0$, where the coefficients $\alpha_i, \beta_i$ are not all 0.

Since we assume not all coefficients 0, there is some $i$ such that either $\alpha_i \neq 0$ or $\beta_i \neq 0$. If some $\alpha_i \neq 0$, then we must also have some $\beta_j \neq 0$ or $\alpha_1u_1 + \cdots + \alpha_nu_n = 0$ would violate the linear independence of $\{u_1, \ldots, u_n\}$. Similarly, if some $\beta_i \neq 0$, then we must have some $\alpha_j \neq 0$. 

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Thus, we have $\alpha_1 u_1 + \cdots + \alpha_n u_n = -(\beta_1 w_1 + \cdots + \beta_m w_m)$, where both sides of this equation are nonzero. However, then we have found some nonzero element in both $W$ and $U$. Given that $U \cap W = 0$, this is a contradiction. We cannot have $\alpha_1 u_1 + \cdots + \alpha_n u_n + \beta_1 w_1 + \cdots + \beta_m w_m = 0$ without all coefficients zero. Thus our set is linearly independent.

Let $u + w \in U + W$ be arbitrary. Then since $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$ and $w = \beta_1 w_1 + \cdots + \beta_m w_m$, we have $u + w = \alpha_1 u_1 + \cdots + \alpha_n u_n + \beta_1 w_1 + \cdots + \beta_m w_m$. Thus our set is spanning and linearly independent, proving the claim.

Now we have by definition that $\dim(U) = n$, $\dim(W) = m$, and $\dim(U + W) = n + m$. Thus $\dim(U + W) = \dim(U) + \dim(W)$ as hoped.

(b) Let $V = \mathbb{R}^2$ over $\mathbb{R}$. Then consider the subspaces $U = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $W = \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $X = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

It is clear that $U + W + X = \mathbb{R}^2$ since even $U + W = \mathbb{R}^2$ (An arbitrary $\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$). Thus $\dim(U + W + X) = 2$.

Since the intersection of these subspaces is pairwise trivial, we then have that $\dim(U) + \dim(W) + \dim(X) - \dim(U \cap W) - \dim(U \cap X) - \dim(W \cap X) + \dim(U \cap W \cap X) = 1 + 1 + 1 - 0 - 0 - 0 + 0 = 3$. Thus we see that the equality does not hold.