Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the "extended glossary" on separate paper (\LaTeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please staple this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.

Exercises.

1. Let $F$ be a field, $V$ a vector space over $F$, and $v, x, y \in V$ vectors. Prove that if $v + x = v + y$, then $x = y$.

2. Let $F$ be a field, $V$ a vector space over $F$, $v \in V$ a vector, and $\alpha \in F$ a scalar. Prove that if $\alpha \cdot v = \vec{0}$, then $\alpha = 0$ or $v = \vec{0}$.

3. Show that $\mathbb{C}$ is a vector space over the field $\mathbb{R}$. More generally, if $F \subset K$ are both fields (with addition, multiplication, 0, 1, in $F$ induced from the same operations/elements on $K$), is $K$ a vector space over $F$? (for this one case, you should either provide a counter-example or a one or two line reason, no proof is required this time).

4. Suppose that $V$ and $W$ are vector spaces over a field $F$. We define a new vector space, called the **direct sum** of $V$ and $W$, and denoted by $V \oplus W$, to be the set $V \times W$ of all pairs $\{(v, w) \mid v \in V, w \in W\}$.

   (a) Write down what you think the definition of addition and scalar multiplication should be.

   (b) What is the zero vector in $V \oplus W$?

   (c) Prove that this is a vector space.

5. Which of the following subsets of $F[x]$ are vector spaces over $F$ (as usual, please justify your answers!):

   (a) $V$ is the set of all polynomials $f(x)$ such that $f(2) = 0$. 
(b) $V$ is the set of all polynomials $f(x)$ such that $f''(x) = 1$.

(c) $V$ is the set of polynomials $f(x)$ whose $x^2$ coefficient is zero (i.e. $x^2$ doesn’t appear in $f(x)$).

6. Let $V = \mathbb{Z}/4\mathbb{Z}$ be the integers modulo 4. Is it possible to define the structure of a vector space on $V$ over $\mathbb{F}_2$ so that vector addition is addition modulo 4?

7. Let $\mathbb{R}^+$ denote the positive real numbers. Let

$$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \mathbb{R}^+ \right\},$$

and define vector addition on $V$ by $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ bd \end{pmatrix}$ and scalar multiplication by $e \in \mathbb{R}$ to be $e \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ae \\ be \end{pmatrix}$. Prove that $V$ is a vector space over $\mathbb{R}$.

8. Suppose that $V$ is a vector space, and $U_1, U_2, \ldots, U_n$ are subspaces of $V$. Is $U_1 \cap \cdots \cap U_n$ a subspace? Is $U_1 \cup U_2$ a subspace? Please justify your answers.

**Extended Glossary.** There is no extended glossary this week.

**Journal entry.** Please write a journal entry this week. You can address any of the following, or just tell me about some cool math you’ve been thinking about or want to know more about. Let me know how the course is going. Do you have any concerns with the class so far? Are you finding the resources you need to succeed in the course?

You may work in groups, but please write up your solutions yourself. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems. Each one (example/non-example/theorem) should be included in some group member’s extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.