Math 4310 Homework 10 Solutions

Throughout, let $A \in \text{Mat}_{n \times n}(\mathbb{C})$ have Jordan form $B = J_{\lambda_1, a_1} \oplus \cdots \oplus J_{\lambda_m, a_m}$ ($\lambda_i$ may not be distinct):

We will make use of the following lemmas:

(1) The number of $a_j$ such that $\lambda_j = \lambda_i$ is the dimension of $\ker(A - \lambda_i I)$

pf: Since $QAQ^{-1} = B$, $Q(A - \lambda_i I)Q^{-1} = B - \lambda I$ and so $\dim \ker(A - \lambda_i I) = \dim \ker(B - \lambda_i I)$. So look at the form of $B - \lambda I$: In each Jordan block corresponding to $\lambda_i$ there will be one zero column. The other columns are independent, so the rank of $B - \lambda_i I$ is $n - (\# \lambda_i$ Jordan blocks). The claim follows by rank-nullity.

(2) The sum of $a_j$ such that $\lambda_j = \lambda_i$ is the power of $(x - \lambda_i)$ in characteristic polynomial $f_A$

pf: Recall $f_A = f_B$ since they are similar. Then the sum of $a_j$ for which $\lambda_j = \lambda_i$ is exactly the power of $(x - \lambda_i)$ in $f_B = f_{J_{\lambda_1, a_1}} \cdots f_{J_{\lambda_m, a_m}} = (x - \lambda_1)^{a_1} \cdots (x - \lambda_m)^{a_m}$.

(3) The maximum $a_j$ such that $\lambda_j = \lambda_i$ is the power of $(x - \lambda_i)$ in the minimal polynomial $\mu_A$

pf: Recall $\mu_A = \mu_B$ since they are similar. Then the maximum of $a_j$ such that $\lambda_j = \lambda_i$ is exactly the power of $(x - \lambda_i)$ in $f_B = \text{lcm}(f_{J_{\lambda_1, a_1}}, \cdots, f_{J_{\lambda_m, a_m}}) = \text{lcm}((x - \lambda_1)^{a_1}, \cdots, (x - \lambda_m)^{a_m})$.

1. First, recall permutation matrix $P_{3,5}$ is self-inverse. Let

$$C = P_{3,5} A P_{3,5} = P_{3,5} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From the class notes (and intuitively because $C$ and $A$ are the same linear transformation in different bases), the answers to all of the following questions for $A$ are the answers for $C$:

(a) $f = \det \left(x I - \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right) = \det \begin{pmatrix} x - 1 & 0 & -1 & 0 & 0 \\ 0 & x - 1 & 0 & -1 & 1 \\ 0 & 0 & x - 1 & 0 & -1 \\ 0 & 0 & 0 & x - 1 & 0 \\ 0 & 0 & 0 & 0 & x - 1 \end{pmatrix} = (x - 1)^6 \implies \lambda = 1$

(b) From the notes, \{roots of $\mu$\} = \{roots of $f$\} = \{1\}. So $\mu = (x - 1)^k$. Thus, since $\mu$ generates $\text{ann}(C)$, $\mu = (x - 1)^p$ for the smallest $p$ s.t. $(x - 1)^p \in \text{ann}(C)$:

$$(C - 1 I) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (C - 1 I)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(C - 1 I)^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 0 \implies \mu = (x - 1)^3$$
(c) \( \dim \ker(C - I) = 6 - \dim \im(C - I) = 6 - \rank \left( \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \). Noting the three nonzero rows are independent, we have \( \dim \ker(C - I) = 3 \).

(d) Since \( \dim \ker(C - I) = 3 < 6 \), there is no basis of \( \mathbb{C}^6 \) composed of eigenvectors of \( C \). Thus \( C \) is not diagonalizable.

(e) By (c) and (1), \( B = J_{1,a_1} \oplus J_{1,a_2} \oplus J_{1,a_3} \). By (b) and (3), \( B \) has one Jordan block \( J_{1,3} \).

2. \( A \)'s only eigenvalue is 0, so all \( \lambda_j = 0 \). By (2), \( \sum a_j \) s.t. \( \lambda_j = 0 \) is 4. So either \( (a_1 = 4) \) or \( (a_1 = 3, a_2 = 1) \) or \( (a_1 = 2, a_2 = 1_1, a_3 = 1) \) or \( (a_1 = 2, a_2 = 2) \) or \( (a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1) \) or \( (a_1 = 1, a_2 = 2, a_3 = 3) \) or \( (a_1 = 1, a_2 = 3) \).

3. Now \( \lambda_j = 2 \) or \( \lambda_j = 3 \). By (2) and (3), \( \sum a_j \) s.t. \( \lambda_j = 2 \) is 4 and max \( a_j \) s.t. \( \lambda_j = 2 \) is 2. Similarly, \( \sum a_j \) s.t. \( \lambda_j = 3 \) is 2 and max \( a_j \) s.t. \( \lambda_j = 3 \) is 1. So, in any order, \( B \) will contain:
   - A Jordan block \( J_{3,1} \) and a Jordan block \( J_{2,2} \)
   - Remaining Jordan blocks \( J_{3,1} \) and (either \( J_{2,2} \) or two copies of \( J_{2,1} \))

4. All \( \lambda_j = 13 \). By (2), (3), and (1): \( \sum a_j = 6, \) max \( a_j = 3 \), and there are a total of 3 Jordan blocks. Thus, in any order, \( B \) will contain:
   - A Jordan block \( J_{13,3} \) and remaining two Jordan blocks \( \{ J_{13,1}, J_{13,2} \} \)

5. (a) -Note that \( \ker(T_i) \subset \ker(T_{i+1}) = \ker(T \circ T_i) \). Then \( \dim \ker(T_i) \leq \dim \ker(T_{i+1}) \). Thus, since \( V \) finite dimensional, there must be a \( j \) such that \( \ker T_j = \ker T_i \) for \( i \geq j \) (the dimension cannot go down, nor increase forever!). Claim that \( V \) is a \( (T, j) \) direct sum:
   - Let \( v \in \ker T_j \cap \im T_j \). Then \( T_j^j(v) = 0 \) and \( T_j^j(w) = v \). So \( T_j^j(w) = T_j^i(v) = 0 \) \( \implies \) \( T_j^j(w) = 0 \) by choice of \( j \). Hence \( v = 0 \), so \( \ker T_j \cap \im T_j = \{0\} \).
   - By rank-nullity, \( \dim V = \dim \im(T_j^j) + \dim \ker(T_j^j) \). So the trivial intersection above shows \( V = \im T_j^j \oplus \ker T_j^j \) as hoped.

(b) For example, consider \( W = F[x] \) and \( F : W \rightarrow W \) defined by \( F(\sum_{i=0}^n c_i x^i) = \sum_{i=1}^n c_i x^{i-1} \). Clearly \( F \) linear. Notice that \( \im T_j^j = W \) (taking polynomials of arbitrarily high degree). Since \( F^j \) is never trivial, we will never have \( W \) an \( (F, j) \) direct sum.

6. If \( S \circ T \) diagonalizable, then \( ST \) similar to a diagonal matrix. Note that \( S^{-1}(ST)S = TS \), so \( ST \) is similar to \( TS \). Thus, since similarity an equivalence relation, \( TS \) diagonalizable as well.

7. Consider the \( m \times m \) matrix \( M_m = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Multiplying matrices, we obtain \( M_m M_m = I_m \) (so \( M_m \) self inverse) and \( M_m J_{\lambda,m} M_m = J_{\lambda,m}^T \). Now, consider the block matrix \( M = \begin{bmatrix} B_{a_1} & 0 & \ldots & 0 \\ 0 & B_{a_2} & \ldots & 0 \\ 0 & 0 & \ldots & B_{a_m} \end{bmatrix} \). Again \( MM = I \), and now \( MBM = B^T \).

Now \( A = QBQ^{-1} \implies A^T = (Q^T)^{-1}B^TQ^T = (Q^T)^{-1}(MBM)Q^T = (Q^T)^{-1}M(Q^{-1}AQ)MQ^T = (QM(Q^T)^{-1})A(QM(Q^T)^{-1}) \), showing \( A \) similar to \( A^T \).

8. The problem with the proof is that \( A^T = -A \iff J^T = (QAQ^{-1})^T = -QAQ^{-1} = -J \). For example, \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) is skew symmetric but nonzero.