Please print out these pages. I encourage you to work with your classmates on this homework. Please list your collaborators on this cover sheet. (Your grade will not be affected.) Even if you work in a group, you should write up your solutions yourself! You should include all computational details, and proofs should be carefully written with full details. As always, please write neatly and legibly.

Please follow the instructions for the “extended glossary” on separate paper (\TeX it if you can!) Hand in your final draft, including full explanations and write your glossary in complete, mathematically and grammatically correct sentences. Your answers will be assessed for style and accuracy.

Please staple this cover sheet, your exercise solutions, and your glossary together, in that order, and hand in your homework in class.

Exercises.

1. Consider the following $6 \times 6$ matrix over $\mathbb{C}$.

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

(a) Find the characteristic polynomial and eigenvalues of $A$.
(b) Find the minimal polynomial of $A$.
(c) Find the dimensions of the eigenspaces of $A$.
(d) Is $A$ diagonalizable?
(e) Find the Jordan canonical form of $A$.

2. Find all possible Jordan canonical forms for a $4 \times 4$ matrix $A$ such that the characteristic polynomial is $f_A(x) = x^4$.

3. Find all possible Jordan canonical forms for a $6 \times 6$ matrix $A$ such that the characteristic polynomial is $f_A(x) = (x-2)^4(x-3)^2$ and the minimal polynomial is $\mu_A(x) = (x-2)^2(x-3)$.

4. Consider all $6 \times 6$ matrices $A$ which have the following properties:
(a) $f_A(x) = (x - 13)^6$, and
(b) $\mu_A(x) = (x - 13)^3$, and
(c) The eigenspace $\ker(A - 13I)$ has dimension 3.

What are the possible Jordan canonical forms of $A$?

5. Let $V$ be a vector space, $T : V \to V$ a linear transformation and $k \ge 1$ a natural number. Recall that $T^k$ denotes the $k$-fold composition $T \circ T \circ \cdots \circ T$. We will say that $V$ is a $(T, k)$-direct sum if $V = \text{im}(T^k) \oplus \ker(T^k)$.

(a) Let $V$ be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Show that there exists an integer $k \ge 1$ such that $V$ is a $(T, k)$-direct sum.

(HINT: To get started, consider the relationship between $\ker(T^i)$ and $\ker(T^{i+1})$.)

(b) Give an example of an infinite dimensional vector space $W$ and a linear transformation $F : W \to W$ such that $W$ is not an $(F, k)$-direct sum for any natural number $k \ge 1$.

6. Let $S, T \in L(C^n, C^n)$ with $S$ invertible. Assume that $S \circ T$ is diagonalizable. Prove that $T \circ S$ is diagonalizable.

7. Prove that any matrix $A \in M_n(C)$ is similar to its transpose $A^T$. (Warning: If you think you have a way of proving this without working with the Jordan canonical form, it’s probably wrong.)

8. What is wrong with the following proof? If $A$ is a complex $n \times n$ matrix such that $A^T = -A$ ($A$ is called skew-symmetric), then $A = 0$. Proof: Let $J$ be the Jordan normal form of $A$. Since $A^T = -A$, then $J^T = -J$. But $J$ is triangular, so that $J^T = -J$ implies that every entry of $J$ is zero. Since $J$ is zero, and $A$ is similar to $J$, then $A = 0$. (Also give a counter-example to this theorem!)

9. Let $C^0([0, 1], R)$ denote continuous $R$-valued functions on the interval $[0, 1]$, equipped with the inner product

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x) \, dx.$$ 

This has a three-dimensional subspace $W = \text{Span}([1], x, x^2)$.

(a) Find an orthogonal basis for $W$, and then find an orthonormal basis for $W$.
(b) Find $\text{pr}_W(x^3)$.
(c) Find $\text{pr}_W(\sqrt{x})$.

These projections are the best approximations to the original functions that are in the subspace $W$. “Best” means closest in terms of the norm defined by this inner product.

Extended Glossary. There is no extended glossary this week.

Journal entry. There is no journal entry this week.

You may work in groups, but please write up your solutions yourself. If you do work together, your group should come up with at least two examples, two non-examples, and two theorems.
Each one (example/non-example/theorem) should be included in some group member’s extended glossary. Your solutions should be written formally, so that we could cut and paste them into a textbook.