1. Short Answer

(a) i. $U_1$ is not a subspace: Take $T_1(v) = v$ and $T_2(v) = -v$. Both are injective but their sum is $(T_1 + T_2)(v) = 0$.

ii. $U_2$ is a subspace: $0(t) = 0(1/(1 + t^2))$ so $0 \in U_2$. For all $t, c \in \mathbb{R}$ and $f, g \in U_2$, $(f + g)(t) = f(t) + g(t) = (f + g)/(1 + t^2)$ and $(cf)(t) = cf(t) = cf)/(1 + t^2))$. 

(b) i. $T_1$ is not linear: $T_1(a f(x)) = a f'(x) - r \neq aT_1(f(x))$.

ii. $T_2$ is linear: $T_2(p(x) + q(x)) = p'(x) + q'(x) - (p(x^2) + q(x^2)) = T_2(p(x)) + T_2(q(x))$ and $T_2(ap(x)) = ap'(x) - ap(x^2) = aT_2(p(x))$.

(c) i. The dimension is 7: Take standard basis $\{e_1, \ldots, e_7\}$.

ii. The dimension is 8: Take basis $\{e_1, \ldots, e_4, [i, 0, 0, 0], [0, i, 0, 0], [0, 0, i, 0], [0, 0, 0, i]\}$.

iii. The dimension is $\infty$: Consider infinite linear independent set $\{f_n | n \in \mathbb{N}, f_i(i) = 1$ and $f_i(j) = 0 \text{ else}\}$. 

iv. The dimension is 5: Take basis $\{f_1, \ldots, f_5 \mid f_i(i) = 1$ and $f_i(j) = 0 \text{ else}\}$.

(d) i. True, since $a \mathbb{Z} + b \mathbb{Z} + c \mathbb{Z} = (a \mathbb{Z} + b \mathbb{Z}) + c \mathbb{Z}$, we can take $m = \gcd(\gcd(a, b), c)$.

ii. False, take $\{(0, 1), [0, 2], [1, 0]\} \subset \mathbb{R}^2$

iii. False, take the trivial subspace $\{0\}$ (note $\{0\}$ is not linearly independent, $a0 = 0$ for $a \neq 0$).

iv. False, if $k$ vectors span $V$ then $V$ has a basis of size $\leq k$.

2. (a) Consider an arbitrary element $v + W \in V/W$. Using the basis of $V$, $v + W = (a_1 v_1 + \cdots + a_m v_m) + a_{m+1} v_{m+1} + \cdots + a_n v_n) + W = (w + a_{m+1} v_{m+1} + \cdots + a_n v_n) + W$ for some $w \in W$. Thus $v + W = (a_{m+1} v_{m+1} + \cdots + a_n v_n) + W = a_{m+1} (v_{m+1} + W) + \cdots + a_n (v_n + W)$, so $(v_{m+1} + W, \ldots, v_n + W)$ spans $V/W$.

Now let $a_{m+1} (v_{m+1} + W) + \cdots + a_n (v_n + W) = 0 + W$. Then $(a_{m+1} v_{m+1} + \cdots + a_n v_n) + W = 0 + W$ and so $a_{m+1} v_{m+1} + \cdots + a_n v_n = w$ for some $w \in W$. However, then $a_{m+1} v_{m+1} + \cdots + a_n v_n = \sum a_i v_i \implies a_{m+1} v_{m+1} + \cdots + a_n v_n - b_1 v_1 - \cdots - b_m v_m = 0 \implies a_i, b_i = 0$ for all $i$.

So $(v_{m+1} + W, \ldots, v_n + W)$ is linearly independent, and hence is a basis for $V/W$. So $\dim(V/W) = n - m = \dim V - \dim(W)$ as hoped.

(b) Claim: $B = \left\{ \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right) \right\}$ is a basis for $W$.

Proof:
Let $w = \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \in W$. Then $w = \left( \begin{array}{c} a \\ b \\ -a-b \end{array} \right) = a \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) + b \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + c \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right)$, so the set is spanning.

Now let $a \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) + b \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + c \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} a \\ b \\ -a-b \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right)$. Then $a = b = c = 0$, so the set is LI. Thus $B$ is a basis for $W$ and $\dim(W) = 3$.

Claim: $B = \left\{ \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right) \right\}$ is a basis for $V/W$.

Proof: Let $v + W = \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) + W \in V/W$. Recalling the basis for $W$, we see $B$ is spanning because
(a) Let \( v + W = (a \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right) + b \left( \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right) + c \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) + (d + a + b) \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) + W = (d + a + b) \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) + W. \)

Now, let \( a \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) + W \) = \( \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) + W = 0 + W. \) Then \( \left( \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \in W \implies 0 + 0 + a = a = 0. \) So \( B \) is linearly independent and forms a basis for \( V/W. \) We have \( \dim(V/W) = 1 \) (expected from (a)).

3. Since \( \text{im}(T) \cap \ker(T) = 0, \) it is enough to show that \( \text{im}(T) \cup \ker(T) = V. \) Recall from homework that \( \text{im}(T) \cap \ker(T) = 0 \implies \dim(\text{im}(T) + \ker(T)) = \dim(\text{im}(T)) + \dim(\ker(T)). \)

Then by rank nullity \( \dim(\text{im}(T) + \ker(T)) = \dim(V). \) Then since \( V \) is finite dimensional, a basis for \( \text{im}(T) + \ker(T) \) must be a basis for \( V. \) Thus we must have \( \text{im}(T) + \ker(T) = V \) as hoped.

4. (a) Let \( a_1 T_1 + \cdots + a_n T_n = 0. \) Then for all \( i \in \{1, \ldots, n\}, \) we have \((a_1 T_1 + \cdots + a_n T_n)(v_i) = a_1 T_1(v_i) + \cdots + a_n T_n(v_i) = a_1 T_1(v_i) = a_i = 0, \) so \((T_1, \ldots, T_n)\) is linearly independent.

Now let \( T \in V^* \) be arbitrary.

Claim: \( T = T(v_1) T_1 + \cdots + T(v_n) T_n, \) hence \((T_1, \ldots, T_n)\) spans \( V^*. \)

Proof: For all \( v_i, \) we have \( T(v_i) = T(v_i) T_1(v_i) = T(v_i) T_1(v_i) + \cdots + T(v_i) T_n(v_i) = T(v_i) T_1 + \cdots + T(v_n) T_n(v_i). \) Thus, for any \( v \in V, \)

\[
T(v) = T(b_1 v_1 + \cdots + b_n v_n) = b_1 T(v_1) + \cdots + b_n T(v_n)
\]

\[
= b_1 T(T(v_1) T_1 + \cdots + T(v_n) T_n)(v_1) + \cdots + b_n (T(v_1) T_1 + \cdots + T(v_n) T_n)(v_n)
\]

\[
= (T(v_1) T_1 + \cdots + T(v_n) T_n)(b_1 v_1 + \cdots + b_n v_n)
\]

Thus \((T_1, \ldots, T_n)\) forms a basis for \( V^* \) and the dimension of \( V^* \) is \( n. \)

(b) Let \( v_1, v_2 \in V \) and \( a \in \mathbb{F} \) be arbitrary. Then for any \( w \in W^* \) we have

\[
S^*(w)(av_1) = w(S(av_1)) = w(a(S(v_1))) = aw(S(v_1)) = aS^*(w)(v_1)
\]

\[
S^*(w)(v_1 + v_2) = w(S(v_1 + v_2)) = w(S(v_1) + S(v_2)) = w(S(v_1)) + w(S(v_2)) = S^*(w)(v_1) + S^*(w)(v_2)
\]

by the linearity of \( S \) and \( w. \) So \( S^*(w) \) is a linear transformation for all \( w \in W^*. \)

Let \( w_1, w_2 \in W^* \) and \( a \in \mathbb{F} \) be arbitrary. Then for all \( v \in V, \)

\[
S^*(aw_1)(v) = (aw_1)(S(v)) = aw_1(S(v)) = aS^*(w_1)(v)
\]

\[
S^*(w_1 + w_2)(v) = (w_1 + w_2)(S(v)) = w_1(S(v)) + w_2(S(v)) = S^*(w_1)(v) + S^*(w_2)(v) = (S^*(w_1) + S^*(w_2))(v)
\]

by the definition of function addition and scaling. So \( S^*(aw_1) = aS^*(w_1) \) and \( S^*(w_1 + w_2) = S^*(w_1) + S^*(w_2), \) showing \( S^* \) is linear.

(c) Claim: \( \ker(S^*) = \{0\} \)

Proof: Let \( w^* \in W^* \) be in \( \ker(S^*). \) Then \( S^*(w^*) = 0 \implies w^*(S(v)) = 0 \) for all \( v \in V. \) But \( \text{im}(S) = W, \) so for all \( w \in W \) we must have \( w^*(w) = 0. \) Thus \( w^* = 0. \)