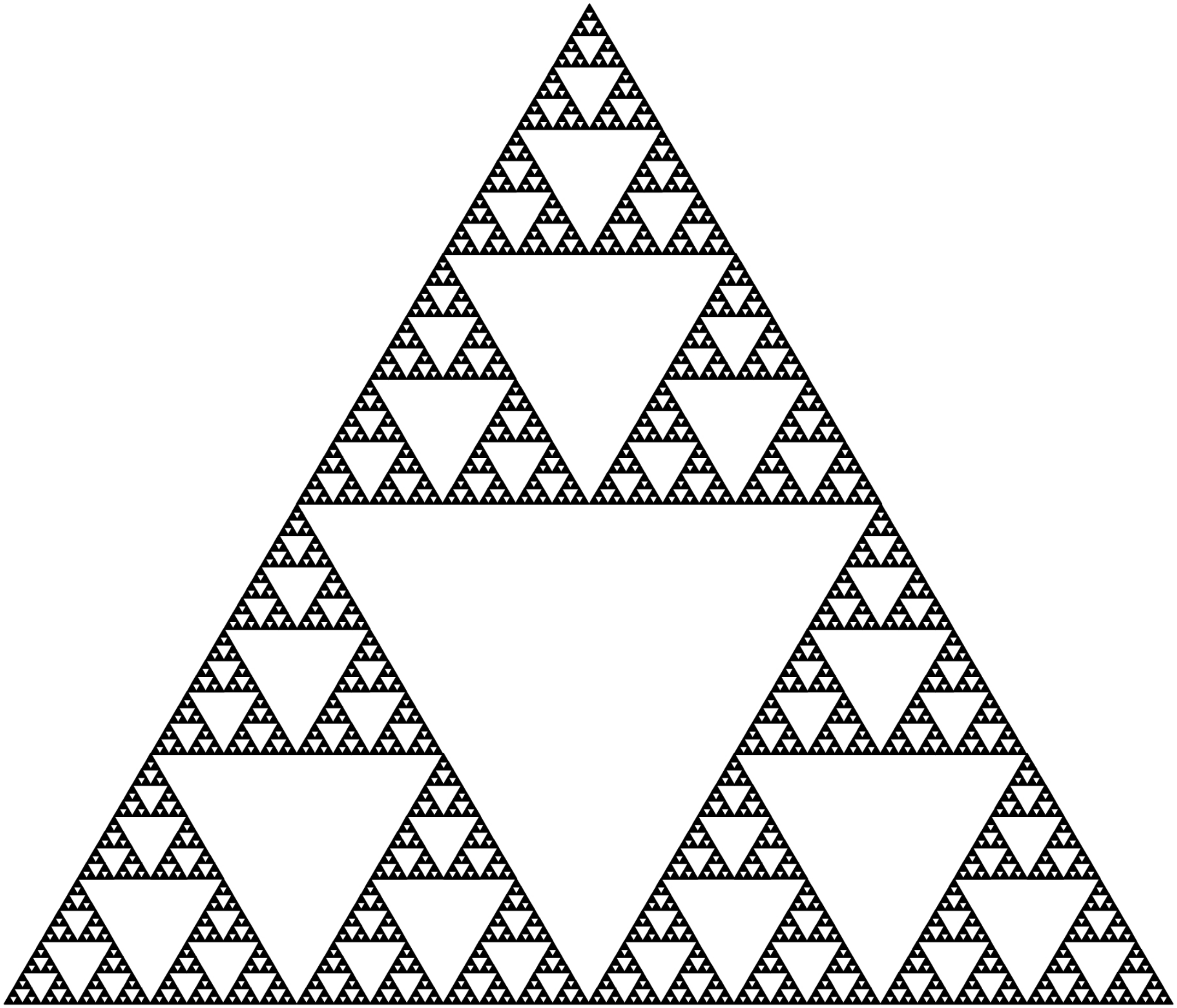
Conformal Energy and Energy Measures on the Sierpinski Gasket (SG)

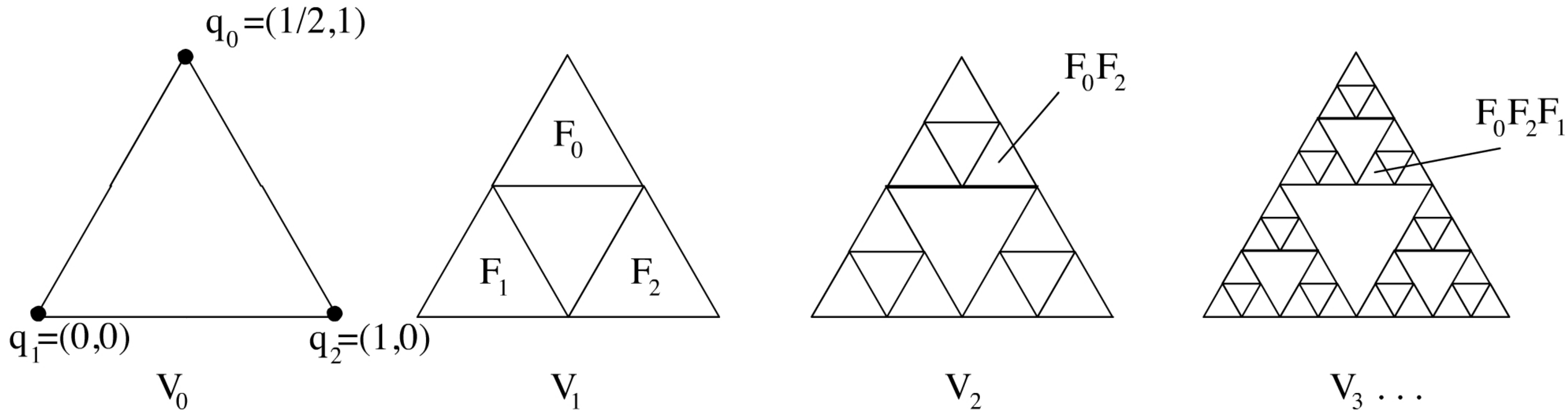
Jonas Azzam

University of Nebraska-Lincoln

July 28, 2005



We analyze the Sierpinski Gasket (K=SG) by looking at a sequence of graphs that converge to SG in the limit:



$$F_i(x) = \frac{1}{2}(q_i - x) + q_i.$$

The cells of a fractal K are $F_w K$ where $w = w_1 w_2 ... w_n$, $w_i \in \{0, 1, 2\}$, and $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m} K$.

For a function f defined on V_m, we define the *level m energy*,

$$E_m(f) = \sum_{x \underset{m}{\sim} y} (f(x) - f(y))^2.$$

A function u on SG is said to be *harmonic* if it minimizes energy on V_m.

If we define an arbitrary function u on V_m, it's *harmonic extension* to V_{m+1} will be $\widetilde{u}$ where $\widetilde{u}\,|_{V_m} = u$, and $E_{m+1}(\widetilde{u}) \leq E_{m+1}(u')$ for all u' on V_m such that $u'|_{V_m} = u$.

In the case of SG, if $\widetilde{u}$ is the harmonic extension of u, then

$$E_{m+1}(\widetilde{u}) = rE_m(u)$$

where $r = \frac{3}{5}$.

We then define the *renormalized energy*

$$\mathcal{E}_m(u) = r^{-m}E_m$$

so that

$$\mathcal{E}_{m+1}(\widetilde{u}) = r^{-m-1}E_{m+1}(\widetilde{u}) = r^{-m-1}rE_m(u) = \mathcal{E}_m(u).$$

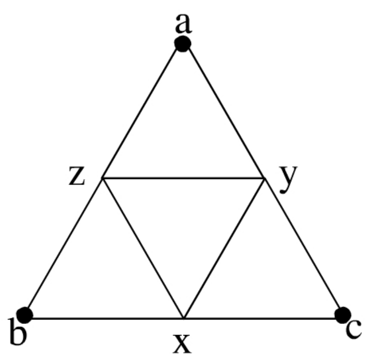
Therefore, for a function defined on V^*, we know

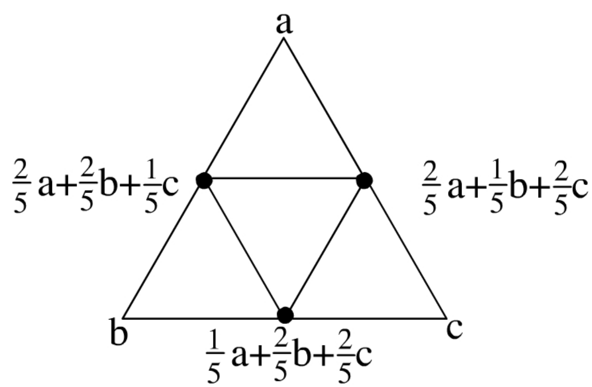
$$\mathcal{E}_m(u) \leq \mathcal{E}_{m+1}(u).$$

We define $\mathcal{E}(u) = \lim_{m\to\infty} \mathcal{E}_m(u)$.

If h is harmonic on V_m, then $\mathcal{E}(h) = \mathcal{E}_m(h) = \mathcal{E}_0(h)$.

In fact, a harmonic function is uniquely determined by its boundary points in V_0.





By differentiating $\mathcal{E}_1(h)$ in terms of x, y, z, and setting the derivatives equal to zero, we get

$$4x - y - z = b + c$$
$$4y - x - z = a + c$$
$$4z - x - y = a + b.$$

Let H_0 be the set of harmonic functions on SG. Then H_0 is a three dimensional vector space. Furthermore, $\mathcal{E}$ is a norm on the two dimensional space $H_0/constants$. In fact, if we define

$$\mathcal{E}_m(u,v) = r^{-m} \sum_{x \sim y} (u(x) - u(y))(v(x) - v(y)),$$

then $\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u,v)$ is an inner product on $H_0/constants$.

Given $h \in H_0$, we define the energy measure

$$\nu_h(F_w K) = r^{-|w|}\mathcal{E}(h \circ F_w).$$

For $h_1, h_2 \in H_0/constants$ such that $\mathcal{E}(u, v) = 0$ and $\mathcal{E}(u) = \mathcal{E}(v) = 1$, the *Kusuoka measure* is

$$\nu = \nu_{h_1} + \nu_{h_2}.$$

Fact: All energy measures are absolutely continuous with respect to ν.

Conformal Energy:

Note that

$$\mathcal{E}(u) = \int_K 1 d\nu_u.$$

Let φ be a function on SG and define a *conformal energy*

$$\mathcal{E}_\varphi(u) = \int_K \varphi d\nu_u.$$

Example: If $\varphi = \sum \alpha_j \chi_{C_j}$ where the C_j are 'disjoint' cells such that $\bigcup C_j = K$, then

$$\mathcal{E}_\varphi(u) = \sum \alpha_j \mathcal{E}_{C_j}(u).$$

For an electrical network Γ, if each edge has conductance $c_{x,y}$ or resistance $r_{x,y} = \frac{1}{c_{x,y}}$, we have an associated energy

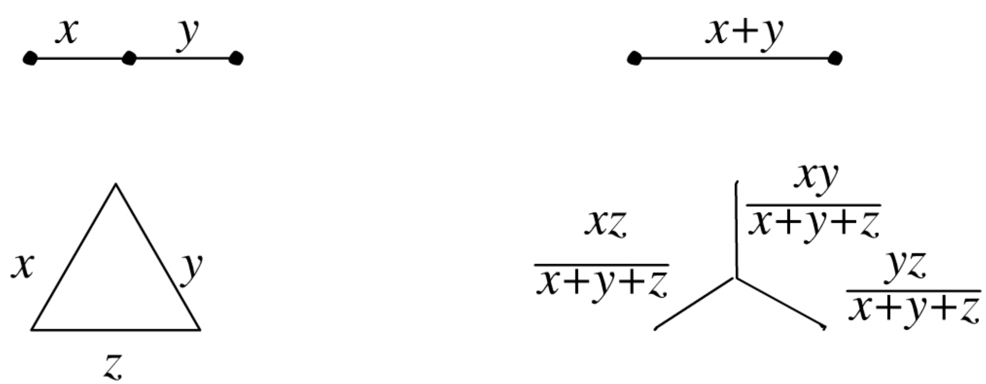
$$\mathcal{E}_\Gamma(u) = \sum_{x \sim y} c_{x,y}(u(x) - u(y))^2.$$

In the case of V_m, the edge resistances are all r^m (and conductances r^{-m}).

We can think of a conformal energy (with a simple function φ) as an energy with particular resistances on the edges of each cell.

Resistances:

Given a network Γ, we can apply different transformations on the edges and corresponding resistances that preserve the energy of the system.

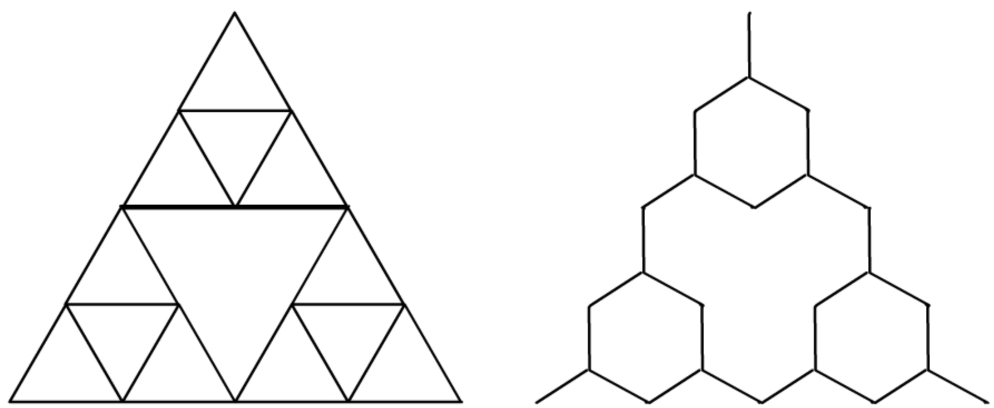


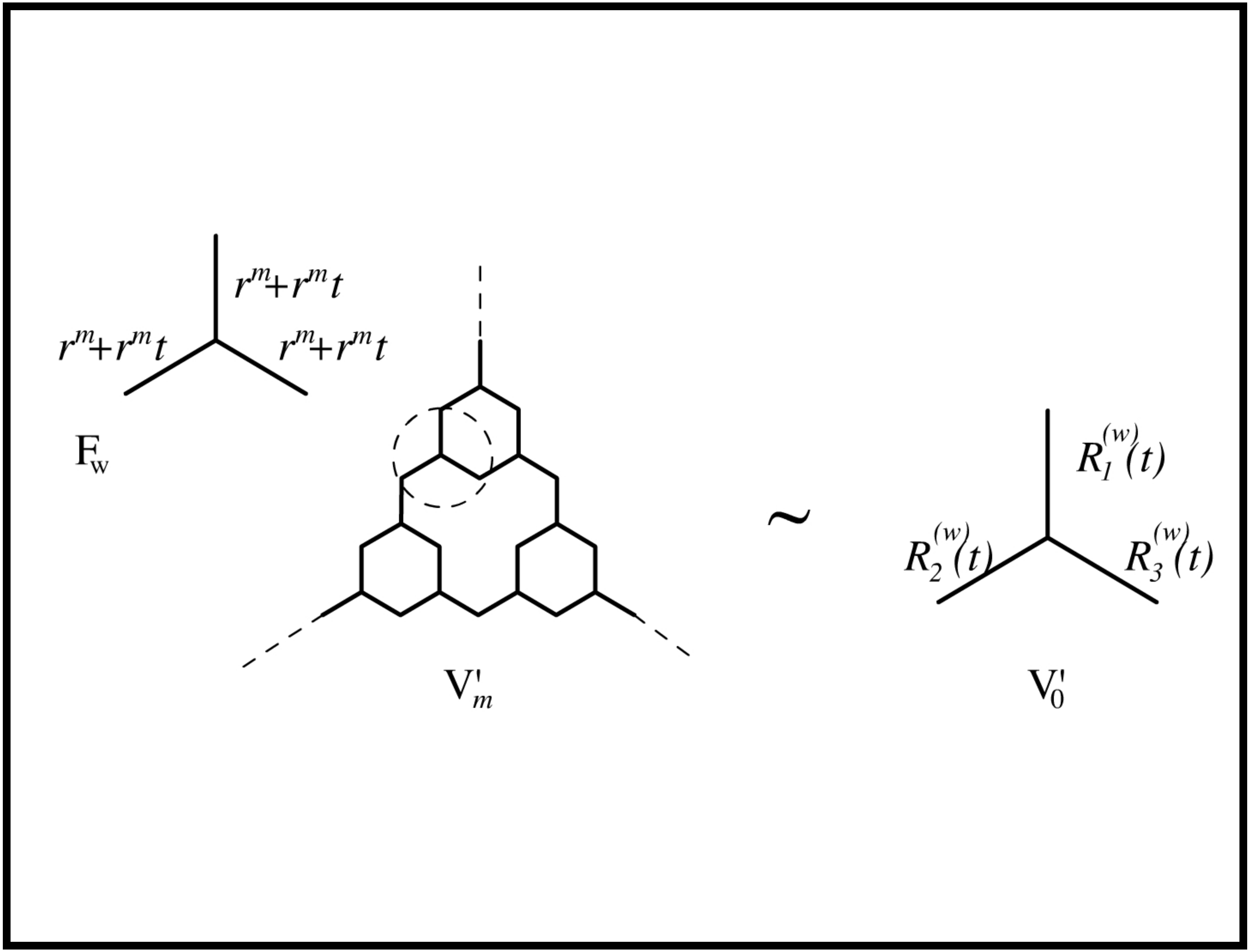
We can use these to compute the restriction of the graph to the boundary points.

Example: If V_m has edge resistances r^m, then the restriction of V_m to V_0 has edge resistances 1.

We looked at the derivative of the resistances in V_0 with respect to changing the resistances of a cell in V_m.

Instead of V_m, look at V'_m after doing Δ-Y transformations on each cell.




$r^m + r^m t$
$r^m + r^m t$
$r^m + r^m t$
F_w
V'_m
~
$R_1^{(w)}(t)$
$R_2^{(w)}(t)$
$R_3^{(w)}(t)$
V'_0

What we found:

1. There exist matrices D_0, D_1, D_2 such that, if we perturb an F_w cell of V_m, then

$$\mu_i(F_w K) = R_i'(0) = e_i^T D_{w_1} D_{w_2} \cdots D_{w_m} \left(1\ 1\ 1\right)^T .$$

2. μ_i is a measure on SG,

3. $\mu_1 + \mu_2 + \mu_3 = \frac{3}{2}\nu$.

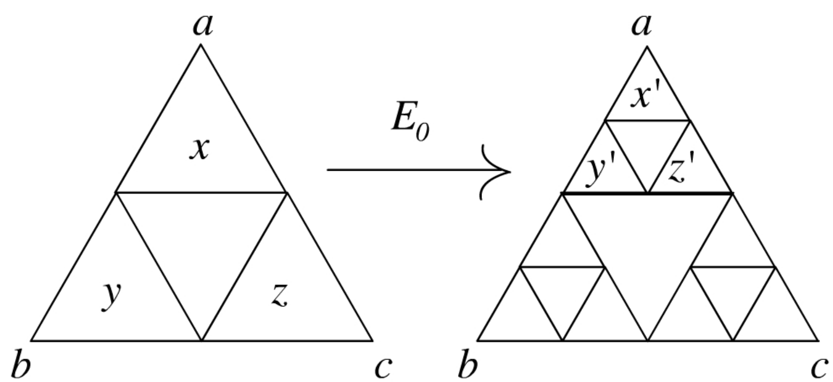
This gives us a nice way of computing the Kusuoka measure, namely

$$\nu(F_w K) = r^{|w|}\,(1\ 1\ 1)\,D_w\,(1\ 1\ 1)^T\ .$$

Linear-like Energy Distribution:

If such a relation exists for the Kusuoka measure, why not for general energy measures?

There exist linear transformations E_i that take these energies to level 2 energies.



$$\mathcal{E}(h \circ F_w) = (1\ 1\ 1)\, E_w\, (x\ y\ z)^T \,.$$

What good is this?

1. Gives us a nice way of expressing the energy measure,

$$\nu_h(F_w K) = r^{-|w|}\,(1\ 1\ 1)\,E_w\,(x\ y\ z)^T\,.$$

2. Helps us calculate the L^p dimensions of energy measures, where

$$\dim{}_p\nu_h = \lim_{m\to\infty} \frac{\log \sum_{|w|=m} \nu_h(F_w K)^p}{(p-1) m \log \frac{3}{5}}.$$

$$\dim{}_2\nu_h = \frac{\log \frac{25}{11}}{\log \frac{5}{3}} \approx 1.6071639985...$$

$$\dim{}_3\nu_h = \frac{\log\left(\frac{31}{225} + \frac{151730163445790^{1/2}}{134217728}\right)}{2\log\frac{3}{5}} \approx 1.4404335708...$$

$$\dim{}_4\nu_h = \frac{\log\left(\frac{1327}{16875} + \frac{3242319174104421^{1/2}}{1073741824}\right)}{3\log\frac{3}{5}} \approx 1.3230040245...$$