Conformal Energy and Energy Measures on the Sierpinski Gasket (SG)

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We analyze the Sierpinski Gasket (K=SG) by looking at a sequence of graphs that converge to SG in the limit:

\[ F_i(x) = \frac{1}{2}(q_i - x) + q_i. \]

The cells of a fractal \( K \) are \( F_w K \) where \( w = w_1 w_2 \ldots w_n, w_i \in \{0, 1, 2\}, \) and \( F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m} K. \)
For a function $f$ defined on $V_m$, we define the *level m energy*,

$$E_m(f) = \sum_{x \sim y}^m (f(x) - f(y))^2.$$ 

A function $u$ on SG is said to be *harmonic* if it minimizes energy on $V_m$.

If we define an arbitrary function $u$ on $V_m$, its *harmonic extension* to $V_{m+1}$ will be $\tilde{u}$ where $\tilde{u} |_{V_m} = u$, and $E_{m+1}(\tilde{u}) \leq E_{m+1}(u')$ for all $u'$ on $V_m$ such that $u' |_{V_m} = u$. 

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In the case of SG, if \( \tilde{u} \) is the harmonic extension of \( u \), then

\[
E_{m+1}(\tilde{u}) = r E_m(u)
\]

where \( r = \frac{3}{5} \).

We then define the renormalized energy

\[
\mathcal{E}_m(u) = r^{-m} E_m
\]

so that

\[
\mathcal{E}_{m+1}(\tilde{u}) = r^{-m-1} E_{m+1}(\tilde{u}) = r^{-m-1} r E_m(u) = \mathcal{E}_m(u).
\]
Therefore, for a function defined on $V^*$, we know

$$\mathcal{E}_m(u) \leq \mathcal{E}_{m+1}(u).$$

We define $\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u)$.

If $h$ is harmonic on $V_m$, then $\mathcal{E}(h) = \mathcal{E}_m(h) = \mathcal{E}_0(h)$. 
In fact, a harmonic function is uniquely determined by its boundary points in $V_0$.

By differentiating $E_1(h)$ in terms of $x, y, z$, and setting the derivatives equal to zero, we get

$$4x - y - z = b + c$$
$$4y - x - z = a + c$$
$$4z - x - y = a + b.$$
Let $H_0$ be the set of harmonic functions on SG. Then $H_0$ is a three dimensional vector space. Furthermore, $\mathcal{E}$ is a norm on the two dimensional space $H_0/\text{constants}$. In fact, if we define

$$\mathcal{E}_m(u, v) = r^{-m} \sum_{x \sim y} (u(x) - u(y))(v(x) - v(y)),$$

then $\mathcal{E}(u, v) = \lim_{m \to \infty} \mathcal{E}_m(u, v)$ is an inner product on $H_0/\text{constants}$. 


Given $h \in H_0$, we define the energy measure

$$\nu_h(F_wK) = r^{-|w|}E(h \circ F_w).$$

For $h_1, h_2 \in H_0/\text{constants}$ such that $E(u, v) = 0$ and $E(u) = E(v) = 1$, the Kusuoka measure is

$$\nu = \nu_{h_1} + \nu_{h_2}.$$

Fact: All energy measures are absolutely continuous with respect to $\nu$. 


Conformal Energy:

Note that

\[ E(u) = \int_K 1d\nu_u. \]

Let \( \varphi \) be a function on \( SG \) and define a conformal energy

\[ E_\varphi(u) = \int_K \varphi d\nu_u. \]

Example: If \( \varphi = \sum \alpha_j \chi_{C_j} \) where the \( C_j \) are 'disjoint' cells such that \( \bigcup C_j = K \), then

\[ E_\varphi(u) = \sum \alpha_j E_{C_j}(u). \]
For an electrical network $\Gamma$, if each edge has conductance $c_{x,y}$ or resistance $r_{x,y} = \frac{1}{c_{x,y}}$, we have an associated energy

$$\mathcal{E}_\Gamma(u) = \sum_{x \sim y} c_{x,y} (u(x) - u(y))^2.$$  

In the case of $V_m$, the edge resistances are all $r^m$ (and conductances $r^{-m}$).

We can think of a conformal energy (with a simple function $\varphi$) as an energy with particular resistances on the edges of each cell.
Resistances:

Given a network $\Gamma$, we can apply different transformations on the edges and corresponding resistances that preserve the energy of the system.

\[ x \quad y \]

\[ x+y \]

\[ x \quad y \quad z \]

\[ x+y+z \]

\[ xz \]

\[ x+y+z \]

\[ xy \]

\[ x+y+z \]

\[ yz \]

\[ x+y+z \]
We can use these to compute the restriction of the graph to the boundary points.

Example: If $V_m$ has edge resistances $r^m$, then the restriction of $V_m$ to $V_0$ has edge resistances 1.
We looked at the derivative of the resistances in $V_0$ with respect to changing the resistances of a cell in $V_m$.

Instead of $V_m$, look at $V_m'$ after doing $\Delta$-$Y$ transformations on each cell.
What we found:

1. There exist matrices $D_0, D_1, D_2$ such that, if we perturb an $F_w$ cell of $V_m$, then

$$\mu_i(F_wK) = R'_i(0) = e_i^T D_{w_1} D_{w_2} \cdots D_{w_m} (1 1 1)^T.$$
2. $\mu_i$ is a measure on SG,
3. $\mu_1 + \mu_2 + \mu_3 = \frac{3}{2}\nu$.

This gives us a nice way of computing the Kusuoka measure, namely

$$\nu(F_w K) = r^{|w|} (1 1 1) D_w (1 1 1)^T.$$
Linear-like Energy Distribution:

If such a relation exists for the Kusuoka measure, why not for general energy measures?
There exist linear transformations $E_i$ that take these energies to level 2 energies.

\[ \mathcal{E}(h \circ F_w) = (1 \ 1 \ 1) \ E_w \ (x \ y \ z)^T. \]
What good is this?

1. Gives us a nice way of expressing the energy measure,

\[ \nu_h(F_wK) = r^{-|w|} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} E_w(x y z)^T. \]

2. Helps us calculate the $L^p$ dimensions of energy measures, where

\[
\dim_p \nu_h = \lim_{m \to \infty} \log \frac{\sum_{|w|=m} \nu_h(F_wK)^p}{(p-1)m \log \frac{3}{5}}.
\]

\[
\dim_2 \nu_h = \frac{\log \frac{25}{11}}{\log \frac{5}{3}} \approx 1.6071639985...
\]

\[
\dim_3 \nu_h = \frac{\log \left( \frac{31}{225} + \frac{151730163445790^{1/2}}{134217728} \right)}{2 \log \frac{3}{5}} \approx 1.4404335708...
\]

\[
\dim_4 \nu_h = \frac{\log \left( \frac{1327}{16875} + \frac{3242319174104421^{1/2}}{1073741824} \right)}{3 \log \frac{3}{5}} \approx 1.3230040245...
\]