Vector Spaces

Determine, with proof, whether each of the following is a vector space. (Note that many of the spaces below are subsets of known vector spaces such as $\mathbb{R}^{\mathbb{R}} := \{f : \mathbb{R} \to \mathbb{R}\}$, so you need only apply the subspace criteria.

- $\emptyset$, the empty set.
- $\{0\}$.
- $\mathbb{N} := \{1, 2, 3, \dots\}$, the natural numbers.
- $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$, the integers.
- $\mathbb{Q} := \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$, the rational numbers.
- $\mathbb{R}$, the real numbers.
- $\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}$, the complex numbers.
- $M_{m\times n}$, the set of $m \times n$ matrices.
- $M^*_{m\times n}$, the set of invertible $m \times n$ matrices.
- $M_{n,\mathbb{Z}}$, the set of $n \times n$ matrices with integer determinant.
- $P := \{f : \mathbb{R} \to \mathbb{R} : f = a_n x^n + \cdots + a_1 x + a_0\}$, the set of polynomials.
- $P_{<\infty} := \{f \in P : f \text{ has finitely many roots}\}$.
- $C := \{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous}\}$, the set of continuous functions.
- $\mathcal{M} := \{f : \mathbb{R} \to \mathbb{R} : x \leq y \Rightarrow f(x) \leq f(y)\}$, the set of increasing functions.
- For a fixed positive integer n, $C^n := \{f : \mathbb{R} \to \mathbb{R} : f \text{ is n times differentiable with } f^{(n)} \text{ continuous}\}$, the set of n-times differentiable functions.
- $C^\infty := \bigcap_{i=1}^{\infty} C^i$, the set of "smooth" or infinitely differentiable functions.
- $C_0^\infty(\mathbb{R}) := \{f \in C^\infty : \exists [a, b] : f(x) \neq 0 \Rightarrow x \in [a, b]\}$, the set of smooth functions with compact support.
- $N(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} f(x)dx = 1\}$, the set of normalized functions.
- $L^1(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} \mid f(x) \mid dx < \infty\}$.
- $L^2(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} f(x)^2 dx < \infty\}$.
- For a fixed positive real number p, $L^p(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} \mid f(x) \mid^p dx < \infty\}$. (Warning: This is a vector space, but its proof is very difficult.)

- $L^\infty(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \exists B :| f(x) | < B \ \forall x\}$, the set of bounded functions.
- $L^\infty_+(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \exists B : f(x) < B \ \forall x\}$, the set of all functions bounded above.
- $\mathcal{O}_\mathbb{R}[-[0]](\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : f(0) = 0\}$, the smooth functions with zeroes at 0.
- For fixed a, b, $I_a^b := \{f : \mathbb{R} \to \mathbb{R} : \int_a^b f(x)dx = 0\}$.

Show that the following functions are linear transformations:

- $D : C^n \to C^{n-1}$ by $D(f) = f'$.
- $E_a : \mathbb{R}^\mathbb{R} \to \mathbb{R}$ by $E_a(f) = f(a)$.
- $\int_a^b : L^1(\mathbb{R}) \to \mathbb{R}$ by $\int_a^b(f) = \int_a^b f(x)dx$.

A linear transformation from V to $\mathbb{R}$ is called a linear functional on V. Prove that V^*, the set of all linear functionals on V, is a vector space. V^* is called the dual space of V.

Let U and V be vector spaces, and let $\mathrm{Hom}_\mathbb{R}(U, V)$ be the set of all linear transformations from U to V. Show that $\mathrm{Hom}_\mathbb{R}(U, V)$ is a vector space.

Let U and V be subspaces of the vector space W. Which of the following are also subspaces?

- $U \cup V$
- $U \cap V$
- $U + V := \{u + v : u \in U, v \in V\}$.