Problems 3, 4, and 7, as well as parts of problem 1, are much more difficult than anything that will be on the test.

1. Determine, with proof, whether each of the following is a vector space. (Note that many of the spaces below are subsets of known vector spaces such as $\mathbb{R}^{\mathbb{R}} := \{f : \mathbb{R} \to \mathbb{R}\}$, so you need only apply the subspace criteria.)

   - $\emptyset$, the empty set.
   - $\{0\}$.
   - $\mathbb{N} := \{1, 2, 3, \dots\}$, the natural numbers.
   - $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$, the integers.
   - $\mathbb{Q} := \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$, the rational numbers.
   - $\mathbb{R}$, the real numbers.
   - $\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}$, the complex numbers.
   - $M_{m \times n}$, the set of $m \times n$ matrices.
   - M_n^*, the set of invertible $n \times n$ matrices.
   - $M_{n,\mathbb{Z}}$, the set of $n \times n$ matrices with integer determinant.
   - $\mathbb{P} := \{f : \mathbb{R} \to \mathbb{R} : f = a_n x^n + \cdots + a_1 x + a_0\}$, the set of polynomials.
   - $L^\infty(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \exists B :\mid f(x) \mid < B \ \forall x\}$, the set of bounded functions.
   - $L_+^\infty(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \exists B : f(x) < B \ \forall x\}$, the set of all functions bounded above.

2. Show that the following functions are linear transformations:

   - $E_a : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$ by $E_a(f) = f(a)$.
   - $\int_a^b : C[a, b] \to \mathbb{R}$ by $\int_a^b(f) = \int_a^b f(x)dx$.

3. A linear transformation from V to $\mathbb{R}$ is called a <u>linear functional on V</u>. Prove that V^*, the set of all linear functionals on V, is a vector space. V^* is called the <u>dual space</u> of V.

4. Let U and V be vector spaces, and let $\mathrm{Hom}_{\mathbb{R}}(U, V)$ be the set of all linear transformations from U to V. Show that $\mathrm{Hom}_{\mathbb{R}}(U, V)$ is a vector space. Which of the spaces above is isomorphic to $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}^n)$?

5. Let U and V be subspaces of the vector space W. Which of the following are also subspaces?

   - $U \cup V$
   - $U \cap V$
   - $U + V := \{u + v : u \in U, v \in V\}$.

6. Let $T : M_{2\times 2} \to \mathbb{R}$ be defined by

$$T(A) = \det A.$$

Is T a linear transformation? Why or why not?

7. Let A be an $n \times (n-1)$ matrix, and define $T : \mathbb{R}^n \to \mathbb{R}$ by

$$T(x) = \det[A|x].$$

Is T a linear transformation? Why or why not? (Hint: consider the cofactor expansion along the last column.)

8. Define the set S by

$$S = \{x \in \mathbb{C} \mid e^x = 1\}.$$

Is S a vector subspace of $\mathbb{C}$? Why or why not?

9. Let $\mathbb{P}^n$ be the real vector space consisting of all polynomials of degree n or lower with real coefficients. Define the map $T : \mathbb{P}^n \to \mathbb{P}^{n-1}$ by $T(p(x)) = p'(x)$.

   (a) Prove that T is a linear transformation.

   (b) Is T is **onto**, why or why not?

   (c) Is T **one to one**, why or why not?

   (d) Prove that $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathbb{P}^n$.

   (e) Using $B_1 = \{1, x, \ldots, x^n\}$ as a basis for $\mathbb{P}^n$ and $B_2 = \{1, x, \ldots, x^{n-1}\}$ as a basis for $\mathbb{P}^{n-1}$, give the matrix which represents T with respect to B_1 and B_2.

   (f) Find a basis for $\mathrm{Ker}(T)$. What is $\dim \mathrm{Ker}(T)$?

   (g) Find a basis for $\mathrm{Im}(T)$. What is $\dim \mathrm{Im}(T)$?

   (h) What is the rank of T?

10. For each matrix below, determine its rank and the dimensions of its nullspace and column space. Find bases for its nullspace and columnspace.

$$A := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} 5 & 5 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

$$D := \begin{pmatrix} 3 & 0 & 0 \\ 3 & 8 & 0 \\ 4 & 7 & 7 \end{pmatrix}, \quad E := \begin{pmatrix} 2 & 3 & 3 & 1 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad F := \begin{pmatrix} 8 & 3 & 3 & 5 \\ 1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

11. For each pair of matrices M and N in the problem above such that the following questions make sense, determine:

(a) The change of coordinates matrix from the basis for $\mathbb{R}^n$ given by the columns of M to the standard basis.

(b) The change of coordinates matrix from the standard basis for $\mathbb{R}^n$ to the basis given by the columns of M.

(c) The change of coordinates matrix from the basis given by the columns of M to the basis given by the columns of N.

(d) The coordinate vector, in terms of the columns of M, for the vector

$$v := \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ or } v := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \text{ or } v := \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

(e) The vector whose coordinate vector, in terms of the columns of M, is

$$[v]_{\mathcal{B}} := \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ or } [v]_{\mathcal{B}} := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \text{ or } [v]_{\mathcal{B}} := \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$