Prove that there are infinitely many primes

Proof: Suppose to the contrary that there are only finitely many primes. Then we can list all the primes $p_1, \dots, p_n$.

Let $a = p_1 \cdots p_n$ be the product of all the primes and let $\alpha = a + 1$.

Now α is greater than or equal to two, so we may write it as a product of primes, $\alpha = p_{i_1} \cdots p_{i_r}$. Let p_j be one of the primes in this decomposition.

Thus there is an integer x such that $\alpha = p_j x$. But, setting y equal to the product of all the other primes, $y = \prod_{i \neq j} p_i$ we have also $a = p_i y$. Thus $1 = \alpha - a = p_i(x - y)$, so 1 is a product of two integers, one of which is greater than 1. This is clearly absurd, so our assumption that there are only finitely many primes must be false.