Math 424 - Assignment 3 - Solutions

Chapter 1, problem 15:

If we consider the periodic extension of \( f \), say \( \tilde{f} \), then \( \tilde{f}(k) = 0 \) if \( k \in \mathbb{Z} \) & \( k \) is odd & \( \tilde{f}(x) = 0 \) in some nbhd of \( k \) identically.

\( \tilde{f}(k) = 1 \) if \( k \in \mathbb{Z} \) & \( k \) is even & \( \tilde{f}(x) = 0 \) in some nbhd of \( k \) identically.

So \( F(x) = f(x) \) except at \( x = \pm \frac{1}{2} \) where \( F(\pm \frac{1}{2}) = \frac{1}{2} \) using Theorem 1.28 (constants are obviously differentiable).

Chapter 1, problem 18:

To see that the previously calculated Fourier series converges uniformly

\[
f(x) = \frac{1}{2} \pi^2 + \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^k \cos(kx)
\]

we could either observe that \( f(x) = x^2 \) on \([-\pi, \pi]\) has a piecewise smooth periodic extension and then use Theorem 1.30 (p. 72) OR more explicitly observe

\[
\sum_{k=1}^{\infty} \frac{14}{k^2} (2)^k = 4 \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty
\]

and then invoke Lemma 1.38 (p. 74)
Now, since we have uniform convergence

\[
\pi^2 = f(\pi T) = \frac{1}{13} \pi^2 + \sum_{k=1}^{\infty} \frac{4}{k^2} (1)^k \cos(k \pi T) = (1)^k
\]

\[
= \frac{1}{13} \pi^2 + \sum_{k=1}^{\infty} \frac{4}{k^2}
\]

\[
\Rightarrow \frac{2}{13} \pi^2 \cdot \frac{1}{4} = \pi^2 \frac{1}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}
\]
Chapter 1, problem 19:  
(Sketch of plots)

(a) 
not uniform

(b) 
not uniform

(c) 
not uniform
(d) \[ \text{Periodic extension of } f \] 
uniform (since \( f \) is piecewise smooth)

(e) 
uniform (\( f \) piecewise smooth)

(f) 
uniform (\( f \) piecewise smooth)
Chapter 1, problem 27:

\[ \| f - g_\delta \|_{L^2}^2 = \int_0^1 |f - g_\delta|^2 \, dx = \int_{\frac{1}{2} - \delta}^{\frac{1}{2} + \delta} \left| f(x) - g_\delta(x) \right|^2 \, dx \]

\[ = \int_{\frac{1}{2} - \delta}^{\frac{1}{2}} 1 \cdot | g_{\frac{1}{2}}(x) |^2 \, dx + \int_{\frac{1}{2}}^{\frac{1}{2} + \delta} 1 - | g_{\delta}(x) |^2 \, dx \]

\[ = \int_{\frac{1}{2} - \delta}^{\frac{1}{2}} \left( \frac{x}{2\delta} - \frac{1}{4\delta} + \frac{1}{2} \right)^2 \, dx + \int_{\frac{1}{2}}^{\frac{1}{2} + \delta} \left( -\frac{x}{2\delta} + \frac{1}{4\delta} + \frac{1}{2} \right)^2 \, dx \]

\[ = \frac{\delta^2}{12} + \frac{\delta}{12} = \frac{\delta}{6} \rightarrow 0 \text{ as } \delta \rightarrow 0 \]

Yes, we can modify \( g_\delta \) to make it \( C^1 \) and still converge in \( L^2 \) to \( f \); e.g.

\[ \tilde{g}_\delta(x) = \begin{cases} 
0 & \text{if } x \in [0, \frac{1}{2} - \delta) \\
\frac{1}{2} \left[ \sin \left( \frac{\pi (x - \frac{1}{2})}{2\delta} \right) + 1 \right] & \text{if } x \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta) \\
1 & \text{if } x \in [\frac{1}{2} + \delta, 1] 
\end{cases} \]

Notice that \( \tilde{g}(x) = \frac{1}{2} \left[ \sin \left( \frac{\pi (x - \frac{1}{2})}{2\delta} \right) + 1 \right] \) looks like

i.e. is a smooth function "connecting" from 0 to 1.
We can e.g. directly verify:

\[ \tilde{g}(\frac{1}{2} - \delta) = 0 \quad \tilde{g}(\frac{1}{2} + \delta) = 1 \]

Also \( \tilde{g}'(x) = \frac{\pi}{4\delta} \cdot \cos\left(\frac{\pi(x - \frac{1}{2})}{2\delta}\right) \)

so that \( \tilde{g}'\left(\frac{1}{2} - \delta\right) = 0 = \tilde{g}'\left(\frac{1}{2} + \delta\right) \) i.e. we "join C".

For convergence, consider direct integration

\[ \| f - \tilde{g} \|_2^2 = \int_{\frac{1}{2} - \delta}^{\frac{1}{2}} (\tilde{g}_0(x))^2 \, dx + \int_{\frac{1}{2}}^{\frac{1}{2} + \delta} (1 - \tilde{g}_0(x))^2 \, dx = \ldots \]

\[ = \left(\frac{3}{8} - \frac{1}{\pi}\right)\delta + \left(\frac{3}{8} + \frac{1}{\pi}\right)\delta \rightarrow 0 \text{ as } \delta \rightarrow 0 \]

**Remark:** I have omitted the details of 2 integrations above, both are easy and involve only a polynomial, respectively a sine and I hope you can do these without problems.

If you can not please talk to me and I can help you on revision.
Chapter 1, problem 28:

(a) Considering a periodic extension, if $f$ is odd, and so only sine terms appear in the Fourier series

\[ b_k = \frac{2}{\pi} \int_0^\pi f(x) \sin(kx) \, dx = \frac{2}{\pi} \int_0^\pi (\pi - x) \sin(kx) \, dx \]

\[ = \frac{2}{\pi} \left[ \int_0^\pi \sin(kx) \, dx - \int_0^\pi x \sin(kx) \, dx \right] \]

\[ = -2 \cdot \frac{1}{k} \cos(kx) \bigg|_0^\pi - \frac{2}{k} \left[ -\frac{1}{k} \cos(kx) x \bigg|_0^\pi - \int_0^\pi \frac{1}{k} \cos(kx) \, dx \right] \]

\[ = -2 \frac{1}{k} \left[ \cos(k\pi) - 1 \right] + \frac{2}{k} \cos(k\pi) = \frac{2}{k} \]

(b) Define

\[ g_N(x) = 2 \sum_{n=1}^N \frac{\sin(nx)}{n} - (\pi - x) \]

(c) \[ g_N'(x) = 2 \left( \sum_{n=1}^N \cos(nx) \right) + 1 \]

\[ = 2\pi \left( \frac{1}{\pi} \left( \frac{1}{2} + \cos x + \cos 2x + \ldots + \cos Nx \right) \right) \]

\[ = 2\pi P_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)} \]

\[ \text{by Lemma 1.23.} \]

(d) \[ g_N'(x) = 0 \quad \text{i.e.} \quad \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)} = 0 \]

iff \[ \sin((N+\frac{1}{2})x) = 0 \quad \text{if} \quad x \neq 0 \quad \text{and} \quad x > 0 \]

the first zero of sine occurs at \( \pi \) to the right of zero so \[ (N+\frac{1}{2})x = \pi \Rightarrow x = \frac{\pi}{(N+\frac{1}{2})} \]
(e) \[ g_N(\Theta) = \int_0^{\Theta} g_N(x) \, dx + g_N(0) \quad \text{by FTC} \]

\[ g_N(\Theta) = \int_0^{\Theta} \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)} \, dx - \pi \]

since \( g_N(0) = -\pi \).

\[ \int_0^{\Theta} \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)} \, dx = \int_0^{\pi} \frac{\sin \phi}{\sin \left( \frac{\phi}{2N+1} \right)} (N+\frac{1}{2})^{-1} \, d\phi \]

Now \( \lim_{N \to \infty} g_N(\Theta) = \lim_{N \to \infty} \int_0^{\pi} \frac{\sin \phi}{\sin \left( \frac{\phi}{2N+1} \right)} \, d\phi - \pi \)

\[ = \lim_{N \to \infty} \int_0^{\frac{\pi}{2}} \frac{\sin \phi}{\phi} \left( \frac{\phi}{\sin \left( \frac{\phi}{2N+1} \right)} \right) \, d\phi - \pi \]

\[ = 2 \int_0^{\pi} \sin \phi / \phi \lim_{N \to \infty} \left( \frac{\phi}{\sin \left( \frac{\phi}{2N+1} \right)} \right) \, d\phi - \pi = 2 \int_0^{\pi} \frac{\sin \phi \, d\phi}{\phi} - \pi \]

\[ = \pi \]

\( \phi \to 0 \) since \( \sin \phi \to 1 \) as \( \phi \to 0 \)

Notice: We have interchanged limits here; this is a non-trivial operation, but in this case, justified. A course in integration theory is going to tell you why.

(g) see attached printout.
Here is the numerical integration for the function in problem 28 (g)

\[ \text{In}[61] := \text{NIntegrate}[2 \cdot \text{Sin}[x]/x, \{x, 0, \pi\}] - \pi \]

\[ \text{Out}[61] = 0.562281 \]

Although it is only one line in this case and you might be justified in not providing such a printout please notice that it is in general good practice to attach a printout to the homework of the code you used in a software like Mathematica, Maple, Matlab et. al.