Math 424 - Assignment 2 - Solutions

Chapter 1, problem 1:

\[ f(x) = x^2, \quad f(-x) = (-x)^2 = x^2 \] so \( f \) is even and we know from Thm. 1.8 that \( b_k = 0 \) \( \forall k \)

\[
\begin{align*}
    a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{3} \pi^2 \\
    a_k &= \frac{2}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx) \, dx = \frac{2}{\pi} \left( \left. x^2 \frac{\sin(kx)}{k} \right|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2x^2}{k} \sin(kx) \, dx \right) \\
    &= -\frac{4}{\pi k} \int_{-\pi}^{\pi} x \sin(kx) \, dx = \frac{4}{\pi k} \left[ \frac{x \cos(kx)}{k} \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{k} \cos(kx) \, dx \\
    &= \frac{4 \pi}{\pi k^2} \cos(k\pi) = \begin{cases} 
    \frac{4k^2}{\pi k^2} & \text{if } k \text{ is even} \\
    -\frac{4k^2}{\pi k^2} & \text{if } k \text{ is odd}
    \end{cases}
\end{align*}
\]

so \( f(x) = \frac{1}{3} \pi^2 + \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^k \cos(kx) \)

This means we can immediately write the partial sums:

\[
\begin{align*}
    S_1 &= \frac{1}{3} \pi^2 - 4 \cos(x) \\
    S_2 &= \frac{1}{3} \pi^2 - 4 \cos(x) + \cos(2x) \\
    S_5 &= \frac{1}{3} \pi^2 - 4 \cos(x) + \cos(2x) - \frac{4}{9} \cos(3x) + \ldots
\end{align*}
\]

e.tc.

For the plots, see next page (it is OK if you use your favourite computer package to generate plots)

We see that the partial sums work well on \([-\pi, \pi]\) and it is also not surprising from the theory that our coefficients from \([-\pi, \pi]\) may fail on \([-2\pi, 2\pi]\).
In (33): \[
\sum_{k=1}^{\infty} \frac{x^2}{k^2} = \pi^2/6
\]

In (46): Now the same graphs for the interval \([-2\pi, 2\pi]\)

P5 = Plot[f2[x, 1], {x, -2\pi, 2\pi}]
P6 = Plot[f2[x, 2], {x, -2\pi, 2\pi}]
P7 = Plot[f2[x, 5], {x, -2\pi, 2\pi}]
P8 = Plot[f2[x, 7], {x, -2\pi, 2\pi}]
Show[GraphicsArray[{P1, P2, P3, P4}]]
Chapter 1, problem 3:

For a cosine series, we consider the even extension of \( f(x) = x^2 \), given by \( f_e(x) = \begin{cases} f(x) & \text{if } x \in [0, \pi] \\ f(-x) & \text{if } x \in [-\pi, 0) \end{cases} \)

The discussion on p. 48 gives:

\[
f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)
\]

\[
a_0 = \frac{1}{\pi} \int_{0}^{\pi} x^2 \, dx = \frac{\pi^2}{3}
\]

\[
a_k = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos(kx) \, dx = \begin{cases} \frac{4}{k^2} & \text{if } k \text{ is even} \\ -\frac{4}{k^2} & \text{if } k \text{ is odd} \end{cases}
\]

So, \( f(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^{k+1} \cos(kx) \)
Chapter 1, problem 4:

Again referring to page 48, we get using an odd extension of \( f(x) = x^2 \), that is, on \([0,1]\),

\[
f(x) = \sum_{k=1}^{\infty} b_k \sin(kt\pi x)
\]

where

\[
b_k = 2 \int_0^1 x^2 \sin(kt\pi x) \, dx = 2 \left[ -\frac{x^2 \cos(kt\pi x)}{k\pi} \right]_0^1 + \left[ \frac{2x \sin(kt\pi x)}{k\pi} \right]_0^1
\]

\[
= -2\frac{x^2 \cos(k\pi) + 4 \int_0^1 x \cos(kt\pi x) \, dx}{k\pi}
\]

Integration by parts

\[
= -2\frac{k\pi \cos(k\pi) + \frac{4}{k^2\pi^2} \int_0^1 x \sin(kt\pi x) \, dx}{k\pi}
\]

\[
= -2\frac{k\pi \cos(k\pi) + \frac{4}{k^3\pi^2} \cos(k\pi) - \left[ -\frac{2}{k\pi} + \frac{4}{k^3\pi^3} \right] \cos(k\pi)}{k\pi}
\]

\[
= \begin{cases} 
  -\frac{2}{k\pi} & \text{if } k \text{ is even} \\
  -\frac{2}{k\pi} - \frac{4}{k^3\pi^3} & \text{if } k \text{ is odd}
\end{cases}
\]

\[
8 \quad f(x) = \sum_{k=1}^{\infty} \left( -\frac{2}{k\pi} + \frac{4}{k^3\pi^3} \right) \cos(k\pi) - \frac{4}{k^3\pi^3}
\]
Chapter 1, problem 7:

\[ f(x) = |\sin x| = 1 - \sin(x) = \sin(x) \] \quad \text{so } f \text{ is even}

Thm 1.8 gives that we just have to look at the cosine terms.

\[ a_0 = \frac{1}{\pi} \int_{0}^{\pi} \sin(x) \, dx = -\frac{1}{\pi} \cos x \bigg|_{0}^{\pi} = \frac{2}{\pi} \]

\[ a_k = \frac{2}{\pi} \int_{0}^{\pi} \sin x \cos kx \, dx = \ldots = \frac{2(1 + \cos k\pi)}{\pi - k^2 \pi} \]

\[ = \begin{cases} 0 & \text{if } k \text{ is odd } (k \neq 1) \\ \frac{4}{\pi - k^2 \pi} & \text{if } k \text{ is even} \end{cases} \]

so \[ f(x) = \frac{2}{\pi} + \sum_{k=2}^{\infty} \left( \frac{2(1 + \cos k\pi)}{\pi - k^2 \pi} \right) \cos(kx) \]

Notice if \( k = 1 \Rightarrow a_1 = \frac{2}{\pi} \int_{0}^{\pi} \sin x \cos x \, dx = \ldots = 0 \]

(*) note that \( \sin(mx + nx) = \sin(mx) \cos(nx) + \cos(mx) \sin(nx) \)

\( \sin(mx - nx) = \sin(mx) \cos(nx) - \cos(mx) \sin(nx) \)

so that \( \sin(mx) \cos(nx) = (\sin((m+n)x) + \sin((m-n)x))/2 \)

which is the only non-trivial step in the integral.
Chapter 1 - problem 11:

\[ f(x) = \cos(x) \text{, we need an odd extension from } [0, \pi] \]

so using the discussion on p. 48

\[
f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)
\]

\[
b_k = \frac{2}{\pi} \int_{0}^{\pi} \cos(x) \sin(kx) \, dx = \ldots = \frac{2k(1 + \cos(k\pi))}{(k^2 - 1)\pi} \quad k \neq 1
\]

again \( b_1 = 0 \)

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Chapter 1 - problem 16:

\[ f(x) \text{ is } \ \overset{f_e(x)}{\rightarrow} \text{ for } x \in [-\pi, \pi] \text{ extend periodically to } \mathbb{R} \]

at \( x_0 = 0 \), pick any sequence \( x_k \to x_0 \) as \( k \to \infty \) then

\[
\lim_{k \to \infty} f_e(x_k) = \lim_{k \to \infty} f(-x_k) = \lim_{k \to \infty} f(\tilde{x}_k) = f(0)
\]

\( \tilde{x}_k = -x_k \), since \( x_k \leq 0 \)

so \( f_e \) is cts at 0. Since \( f_e(x) = f(-x) \) the periodic extension to \( \mathbb{R} \) will also be continuous.

For odd periodic extensions, we can easily find a continuous \( f \) s.t. \( f \) cts on \([0, 1]\) but \( f_0 = \{ f(x) \text{ on } [0,1] \}

\[
\ell f(x) \text{ on } (-1,0]
\]

is not. Simply take \( f(x) = 1 \) on \([0,1]\)

\[
\Rightarrow f_0(x) = -1 \text{ on } (-1,0]
\]

\( \Rightarrow \) discontinuous at 0.
From the fact that for continuity we need

(1) \(-f(0) = f(0)\) we conclude \(f(0) = 0\)

also the same argument holds at \(a\) for \(f(x)\) on \([0, a]\)

so

(2) \(f(a) = 0\)

then the periodic extension will be as if \(f\) is.