Contemporary Mathematics

Computations of spectral radii on $\mathcal{G}$-spaces

Laurent Saloff-Coste and Wolfgang Woess

Dedicated to Professor Toshikazu Sunada on the occasion of his 60th birthday

Abstract. In previous work, we have developped methods to compute norms and spectral radii of transition operators on proper metric spaces. The operators are assumed to be invariant under a locally compact, amenable group which acts with compact quotient.

Here, we present several further applications of those methods. The first concerns a generalization of an identity of Hardy, Littlewood and Pólya. The second is a detailed study of a class of diffusion operators on a homogeneous tree, seen as a 1-complex. Finally, we investigate the implications of our method for computing spectral radii of convolution operators on general locally compact groups and Lie groups.

1. Introduction

In a series of papers [**13**], [**14**], [**15**] based on ideas introduced by Soardi and Woess [**18**] and Salvatori [**16**], we have developed tools to compute the norms and/or spectral radii of Markov operators that are invariant under the left action of a group when the action is either transitive or almost transitive (i.e has a compact factor space). In [**13**], [**14**], we where concerned with countable homogeneous spaces. A typical example is given by the simple random walk on the vertex set of the $(\mathsf{r}+1)$-regular tree $\mathbb{T} = \mathbb{T}_{\mathsf{r}}$. For each vertex v, set $K(v,w) = 1/(\mathsf{r}+1)$ if w is a neighbor of v, and $K(v,w) = 0$ otherwise. There are many groups $\mathcal{G}$ that act transitively on the vertex set of the tree and such that $K(gv, gw) = K(v,w)$ for all $g \in \mathcal{G}$. One such group is the group of those isometries of the tree that fix one end. See Figure 1 below in Section 4. This group is amenable and non-unimodular. These two properties lead to an easy computation of the norms $\sigma_p(K)$ and spectral radii $\rho_p(K)$ of the operator K acting on $L_p(\mathbb{T})$, $1 \le p \le \infty$ (the reference measure here is the counting measure).

In [**15**], the theory developed in [**14**] is extended to general locally compact state spaces. The purpose of the present paper is to illustrate some of the results of [**15**] with concrete applications and examples. In Section 2, we briefly introduce

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the general framework and recall the relevant results of [**15**], in particular the main tool, Theorem 2.1. In the short Section 3, we explain a generalization of Theorem 2.1 and its relation to an identity of Hardy, Littlewood and Pólya [**7**] plus extensions of the latter. The remaining Sections are devoted to two classes of examples: Section 4 deals with diffusion operators with constant coefficients on a regular tree, seen as a 1-complex where each edge is a copy of the unit interval. Section 5 links our results with classic harmonic analysis by looking at convolution operators on semisimple and other Lie groups, as well as general locally compact groups, not always necessarily connected.

2. Theoretical background

The notation used in this section follows closely that of [**15**].

2.A. Invariant measures. Let X be a metric space whose closed balls are compact (i.e., a proper metric space). Let $\mathcal{G}$ be a locally compact group which acts properly on X. Let I be the quotient space $I = \mathcal{G}\backslash X$ (we use this notation because we will always think of $\mathcal{G}$ acting on X from the left). For each $s \in I$, we denote by X_s the corresponding equivalence class in X so that

$$X = \bigcup_{s\in I} X_s\,.$$

Each orbit X_s is a $\mathcal{G}$-homogeneous space, and for each $x \in X_s$ we have $X_s \cong \mathcal{G}/\mathcal{G}_x$ where $\mathcal{G}_x \subset \mathcal{G}$ is the stabilizer of x. By properness of the action, $\mathcal{G}_x$ is compact. On each X_s there is, up to a multiplicative constant, a unique $\mathcal{G}$-invariant measure d_{X_s} . Given a fixed left Haar measure $d_\mathcal{G}$ on $\mathcal{G}$ and a fixed $\mathcal{G}$-invariant measure d_{X_s} on each X_s , we obtain for each $x \in X$ a specific Haar measure $d_{\mathcal{G}_x}$ on the stabilizer $\mathcal{G}_x$ so that

$$\int_\mathcal{G} F(g)\, d_\mathcal{G} g = \int_{X_s} \left(\int_{\mathcal{G}_x} F(g_{x,y}h)\, d_{\mathcal{G}_x} h \right) d_{X_s} y\,. \tag{1}$$

Let

$$\gamma(x) = |\mathcal{G}_x| \tag{2}$$

denote the total mass of the compact group $\mathcal{G}_x$ under $d_{\mathcal{G}_x}$. There is a slight abuse of notation here because the precise normalization of the measure $d_{\mathcal{G}_x}$ depends on the point x. In particular, for two points $x \neq z$ in X, we can have $|\mathcal{G}_x| \neq |\mathcal{G}_z|$ even if $\mathcal{G}_x = \mathcal{G}_z$ as subgroups of $\mathcal{G}$.

We now have to face the question of the choice of a $\mathcal{G}$-invariant measure on X and its decomposition over I. Indeed, as soon as I is not a singleton, there are non-equivalent $\mathcal{G}$-invariant measures on $\mathcal{G}$. Moreover, there are many ways to decompose a given $\mathcal{G}$-invariant measure over the quotient space I.

Let d_X be a $\mathcal{G}$-invariant measure on X. Let $\mathcal{C}_{00}(X)$ be the space of all continuous compactly supported functions on X. Assume that the factor space I is equipped with a measure $d\lambda$, that each orbit X_s is equipped with a fixed $\mathcal{G}$-invariant measure d_{X_s}, and that the measure $d_X x$ decomposes as follows:

(a) For any $f \in \mathcal{C}_{00}(X)$ and its restriction f_s to X_s ,

$$\int_X f(x)\, d_X x = \int_I \int_{X_s} f_s(x)\; d_{X_s} x\; d\lambda(s)\,. \tag{3}$$

(b) The function $\gamma : x \mapsto \gamma(x) = |\mathcal{G}_x|$ induced by the choice of the $\mathcal{G}$-invariant measures d_{X_s} is continuous on X.

We call such a decomposition of d_X a *continuous decomposition.* See [**15**, §9] for a proof of the existence of such a decomposition for any $\mathcal{G}$-invariant measure on X.

2.B. Invariant operators. On X as above, consider a non-negative kernel $k(x, y)$ and the associated operator

$$Kf(x) = \int_X k(x,y)f(y)\, d_X y\,.$$

This definition makes sense, very generally, at least for non-negative functions f. We make the crucial hypothesis that K is $\mathcal{G}$-invariant, that is, for all $x, y \in X$ and $g \in \mathcal{G}$,

$$k(gx, gy) = k(x, y)\,.$$

We do not assume here that k is a Markov kernel although it will be Markovian in most of our applications. Our aim is to compute the norms

$$\sigma_p(K) = \sup\{\|Kf\|_p : f \in L^p(X, dx),\ \ f \geq 0,\ \|f\|_p \leq 1\}$$

and spectral radii

$$\rho_p(K) = \lim_{n\to\infty} \sigma_p(K^n)^{1/n}\,,$$

where $1 \leq p \leq \infty$ and (for $p \neq \infty$)

$$\|f\|_p = \left(\int_X |f(x)|^p\, d_X x\right)^{1/p}\,.$$

Since k is non-negative, this makes sense without further assumption if we admit the possibility that those quantities are infinite.

For each $s, t \in I$ and each $p \in [1, \infty]$, set

$$a_p(s,t) = a_p[K](s,t) = \int_{X_t} \left(\frac{\gamma(y)}{\gamma(x_s)}\right)^{1/p} k(x_s, y)\, d_{X_t} y \tag{4}$$

where x_s is a fixed reference point in X_s and the function γ is defined by (2). By $\mathcal{G}$-invariance, one can check that $a_p(s,t)$ does not depend on the choice of x_s in X_s, see [**15**, Theorem 2.12]. The crucial fact here is that for any $g \in \mathcal{G}$ and $x \in X$, we have (see [**15**, Lemma 2.6])

$$\Delta(g) = \gamma(x)/\gamma(gx)\,, \tag{5}$$

where $\Delta(g)$ is the *modular function* of $\mathcal{G}$ defined by

$$\Delta(g) \int_{\mathcal{G}} f(hg)\, d_{\mathcal{G}} h = \int_{\mathcal{G}} f(h)\, d_{\mathcal{G}} h\,.$$

Denote by $A_p = A_p[K]$ the operator on $L^p(I, d\lambda)$ defined by

$$A_p f(s) = \int_I a_p(s,t) f(t)\, d\lambda(t). \tag{6}$$

We denote by $\sigma_p(A)$ (resp. $\rho_p(A)$) the norm (resp. spectral radius) of A acting on $L^p(I, d\lambda)$. The result whose application we want to illustrate in this paper is the following.

THEOREM 2.1. *Assume that $\mathcal{G}$ is amenable and that we have a continuous decomposition of d_X over I as in* (3)*. Then*

$$\sigma_p(K) = \sigma_p(A_p[K]) \quad \textit{and} \quad \rho_p(K) = \rho_p(A_p[K])\,.$$

When I is compact, this is contained in [**15**, Theorem 5.3]. The proof goes through also when I is non-compact (see also [**14**] for the case when X is countable); compactness of I is needed in [**15**] for the converse, namely, for deducing amenability from the above equalities.

REMARK 2.2. (a) Denote by K^* the operator with kernel $k^*(x,y) = k(y,x)$ (i.e., the "formal" adjoint of K). Then, for any $1 \le p,q \le \infty$ with $1/p + 1/q = 1$, we have $a_q[K^*](s,t) = a_p[K](t,s)$, in accordance with the fact that $\sigma_q(K^*) = \sigma_p(K)$ and $\rho_q(K^*) = \rho_p(K)$.

(b) Let K_1, K_2 be two operators with kernels k_1, k_2 as above. Then, for any $1 \le p \le \infty$, we have

$$a_p[K_1K_2](s,s') = \int_I a_p[K_1](s,t)a_p[K_2](t,s')\,d\lambda(t),$$

that is,

$$A_p[K_1K_2] = A_p[K_1]A_p[K_2]. \tag{7}$$

3. A generalization, and an identity of Hardy, Littlewood and Pólya

Suppose we want to compute the norms and spectral radii of K as above but on $L^p(X, d\mu)$ with $d\mu(x) = m(x)\,d_X x$, where m is a positive and measurable function. Here, we assume furthermore that there is a function $c(\cdot)$ on $\mathcal{G}$ such that

$$m(x)/m(gx) = c(g) \tag{8}$$

for all $x \in X, g \in \mathcal{G}$. A simple argument, using the bijective isometry from $L^p(X,\mu)$ to $L^p(X, d_X)$ given by $f \mapsto m^{-1/p}f$, shows that the norm $\sigma_p(K,\mu)$ and the spectral radius $\rho_p(K,\mu)$ of K acting on $L^p(X,\mu)$ satisfy

$$\sigma_p(K,\mu) = \sigma_p(K_p) \quad \text{and} \quad \rho_p(K,\mu) = \rho_p(K_p)\,,$$

where K_p denotes the operator with kernel

$$\big(m(x)/m(y)\big)^{1/p} k(x,y)$$

acting on $L^p(X, d_X)$. Under the hypothesis (8), the kernel of K_p is invariant under the action of $\mathcal{G}$, which is assumed to be amenable. Theorem 2.1 applies to K_p and gives the following result.

THEOREM 3.1. *Let $\mathcal{G}$, X, I, d_X and λ be as in sections 2A–B. Assume that $\mathcal{G}$ is amenable, and that the $\mathcal{G}$-invariant measure d_X has a continuous decomposition over (I,λ) as in* (3)*. Let k be a non-negative $\mathcal{G}$-invariant kernel on X, and let μ be a measure on X whose density with respect to d_X is positive and satisfies* (8)*.*

Then the associated integral operator K acting on $L^p(X, d\mu)$ satisfies

$$\sigma_p(K,\mu) = \sigma_p(A_{p,\mu}[K]) \quad \textit{and} \quad \rho_p(K,\mu) = \rho_p(A_{p,\mu}[K])\,, \tag{9}$$

where

$$A_{p,\mu}[K](s,t) = \int_{X_t} \left(\frac{\gamma(y)m(x_s)}{\gamma(x_s)m(y)}\right)^{1/p} k(x_s,y)\,d_{X_t}y\,.$$

Above, x_s is again a fixed reference point in X_s, and $\gamma(\cdot)$ is defined by (2).

Let us connect (9) to a well known and useful identity of Hardy, Littlewood and Pólya [**7**]. On $(0,\infty)$, consider a non-negative kernel $k(x,y)$ satisfying $k(tx,ty) = t^{-1}k(x,y)$, i.e., homogeneous of degree -1. Then that identity says that the operator $Kf(x) = \int_0^\infty k(x,y)f(y)\,dy$ is bounded on $L^p((0,\infty),dx)$ if and only if

$$M = \int_0^\infty k(1,y)y^{-1/p}\,dy < \infty\,. \tag{10}$$

Moreover, the associated norm is $\|K\|_{p\to p} = M$. See also Strichartz [**20**] and Stein [**19**, p. 271]. This result is a special case of (9). Indeed, consider $\mathcal{G} = X = (0,\infty)$ as a multiplicative group acting on itself. The action is of course transitive so that I is a singleton. The group $\mathcal{G}$ is amenable and unimodular. The $\mathcal{G}$-invariant measure on $X = (0,\infty)$ is the (multiplicative !) Haar measure $x^{-1}\,dx$. Write $Kf(x) = \int_0^\infty \widetilde{k}(x,y)\,x^{-1}\,dx$ with $\widetilde{k}(x,y) = k(x,y)/y$. Then $\widetilde{k}$ is G-invariant, and the measure $d\mu(x) = dx = m(x)\,x^{-1}\,dx$, where $m(x) = x$, satisfies (8). Now, (9) clearly becomes

$$\|K\|_{p\to p} = \sigma_p(K,\mu) = \int_0^\infty k(1,y)y^{-1/p}\,dy$$

as desired. Section 2 of [**20**] discusses this example and generalizations to higher dimensions. The results presented there can also be derived directly from Theorem 3.1.

4. Diffusions on trees

4.A. The regular tree and its affine group. In this section we illustrate Theorem 2.1 by looking at diffusions on the $(\mathsf{r}+1)$-regular tree viewed as a 1-dimensional simplicial complex.

Let $X = \mathbb{T}$ be the 1-skeleton of the $(\mathsf{r}+1)$-regular tree and let $\mathbb{V} = V\mathbb{T}$ be the vertex set of that tree. Choose a reference vertex o and a reference end (boundary point) ω of $\mathbb{T}$ regarded as a "mythical ancestor". Draw $\mathbb{T}$ in horocycle layers with respect to ω. Call H_0 the horocycle of o. The other horocycles are labeled so that H_n contains the n-th descendant generation of o with respect to ω whereas H_{-n} contains the n-th ascendant generation of o with respect to ω. The tree $\mathbb{T}$ can be parametrized as $\mathbb{T} = \mathbb{V}\times[0\,,\,1)$ where, for any $v\in\mathbb{V}$, $\{v\}\times[0\,,\,1)$ is the oriented edge from v to its (uniquely defined) predecessor. We call this edge the 0-edge at v and we (arbitrarily) enumerate the r remaining edges as $e_i = \{v_i\}\times[0\,,\,1)$, where v_i, $i = 1,\dots,\mathsf{r}$, are the successors of v. The discrete graph metric on $\mathbb{V}$ has an obvious "linear" extension to $\mathbb{T}$. See Figure 1.

For $v\in\mathbb{V}$, we set $\mathfrak{h}(v) = n$ if $v\in H_n$. For $x = (v,t)\in\mathbb{T} = \mathbb{V}\times[0\,,\,1)$, we set

$$\mathfrak{n}(x) = \mathfrak{h}(v) \quad\text{and}\quad \mathfrak{h}(x) = \mathfrak{h}(v) - t\,. \tag{11}$$

The horocycles in $\mathbb{T}$ are the sets $H_s = \{x = (v,t) : \mathfrak{h}(v,t) = s\}$, and if $s = n - t$ with $t\in[0\,,\,1)$ then $H_s = \{x = (v,t) : v\in H_n\}$.

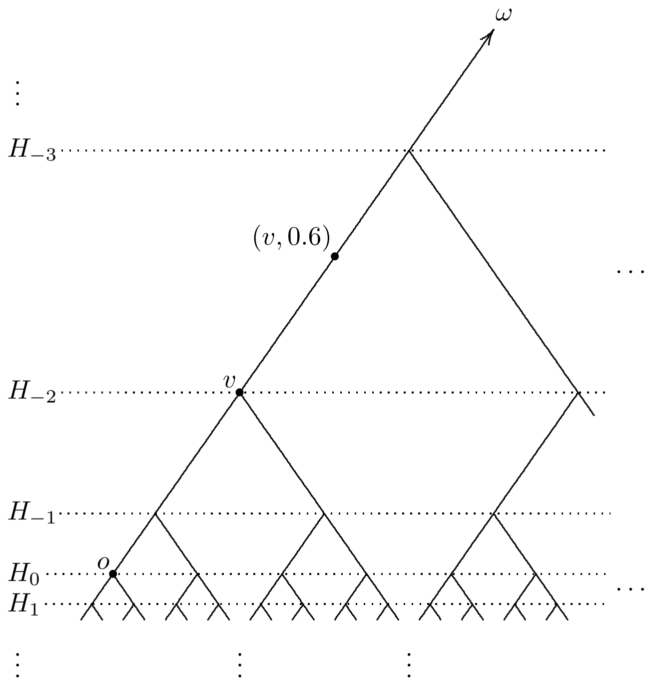

Figure 1

As a reference measure on $\mathbb{T}$, we will use Lebesgue measure on each of the edges, that is the measure $d_{\mathbb{T}}x$ defined by

$$\int_{\mathbb{T}} f(x)\, d_{\mathbb{T}}x = \sum_{v\in\mathbb{V}} \int_0^1 f(v,t)\, dt. \tag{12}$$

Next, we consider the group $\mathcal{G} = \mathsf{Aff}(\mathbb{T})$ of all graph isometries that fix ω, see [**18**], [**13**], [**14**], [**15**]. This group is amenable, it acts properly on $X = \mathbb{T}$, and its action is transitive on $\mathbb{V}$. The factor space $I = \mathcal{G}\backslash\mathbb{T}$ is the circle which we parametrize in an obvious way by the interval $I = [0\,,\,1)$. The natural projection of $(v,t) \in \mathbb{T}$ on I is then simply $t \in I$. The orbits under the action of $\mathcal{G}$ are the sets $X_t = \{(v,t) : v \in \mathbb{V}\}$. The constant multiples of the counting measure on X_t are exactly the measures that are invariant under $\mathcal{G}$.

Let us take the counting measure on $X_0 = \mathbb{V}$ and normalize the left Haar measure on $\mathcal{G}$ by setting $\gamma(o) = |\mathcal{G}_o| = 1$. Then, for any $v \in \mathbb{V}$, $\gamma(v) = |\mathcal{G}_v| = \mathsf{r}^{-\mathfrak{h}(v)}$. By construction, the stabilizer of $(v,t) \in \mathbb{T}$ is $\mathcal{G}_{(v,t)} = \mathcal{G}_v$, but in order to have a continuous decomposition of $d_{\mathbb{T}}$, we should normalize the measure d_{X_t} to be equal to r^{-t} times the counting measure, so that the function of (2) becomes $\gamma(x) = \mathsf{r}^{-\mathfrak{h}(x)}$. Then the measure λ on I is

$$d\lambda(t) = \mathsf{r}^t\, dt\,. \tag{13}$$

With this notation we have found the following.

LEMMA 4.1. *The Lebesgue measure $d_{\mathbb{T}}$ on $\mathbb{T}$ has the continuous decomposition*

$$\int_{\mathbb{T}} f(x)\, d_{\mathbb{T}} x = \int_0^1 \left[\mathsf{r}^{-t} \sum_{v\in\mathbb{V}} f(v,t) \right] d\lambda(t)\,,$$

where the term in brackets is $\int_{X_t} f(x) d_{X_t} x$.

4.B. Diffusions with constant coefficients. The transition operators that we want to study are those associated with the heat semigroups of conservative "constant coefficients" diffusions on $\mathbb{T}$. To define these, we need some notation.

Let f be a function on $\mathbb{T}$. For each vertex v, define f_v to be the function on the open interval $(0\,,\,1)$ given by

$$f_v(t) = f(v,t), \quad t \in (0\,,\,1)\,,$$

and set

$$f_v(0) = \lim_{t\to 0} f_v(t) \quad \text{and} \quad f_v(1) = \lim_{t\to 1} f_v(t)\,,$$

whenever these limits exist. If f_v is continuous on $(0\,,\,1)$ and $f_v(0)$, $f_v(1)$ exist and are finite, we say that f_v is continuous on $[0\,,\,1]$.

Let $d\mu = m(\cdot)\, d_{\mathbb{T}}$ be a measure with a positive density such that m_v is continuous on $[0\,,\,1]$ for each vertex v. Denote by H^1_μ the space of all functions in $L^2(\mathbb{T},\mu)$ whose distributional derivative on each interior edge $v \times (0\,,\,1)$ can be represented by a function $f'_v \in L^2\big((0\,,\,1)\big)$ and such that the function $f' : (v,t) \mapsto f'_v(t)$ (which is defined almost everywhere) is in $L^2(\mathbb{T},\mu)$. Let H^2_μ be the space of those functions f in H^1_μ such that $f' \in H^1_\mu$ and define f'' accordingly. The spaces H^1_μ and H^2_μ are equipped with the norms

$$\|f\|_{H^1_\mu} = \left(\int_{\mathbb{T}} (|f|^2 + |f'|^2)\, d\mu \right)^{1/2} \quad \text{and} \tag{14}$$

$$\|f\|_{H^2_\mu} = \left(\int_{\mathbb{T}} (|f|^2 + |f'|^2 + |f''|^2)\, d\mu \right)^{1/2}, \tag{15}$$

respectively, which turn them into Hilbert spaces. Let us note that any f in H^1_μ can be represented by a function such that each f_v is continuous on $[0\,,\,1]$. Similarly, any $f \in H^2_\mu$ can be represented by a function such that f_v and f'_v are continuous on $[0\,,\,1]$.

We say that a function f is edgewise differentiable if f_v admits a derivative $f'_v(t)$ on each open edge $v \times (0\,,\,1)$. We say that f is edgewise C^1 if it is edgewise differentiable with continuous derivative f'_v on each open edge, and

$$\lim_{t\to 0} f'_v(t) = f'_v(0) \quad \text{and} \quad \lim_{t\to 1} f'_v(t) = f'_v(1)$$

both exist and are finite (i.e., f'_v is continuous on $[0\,,\,1]$). Note that such funcions f need not be continuous at a given vertex v. If f is edgewise differentiable, we set

$$f'(x) = f'_v(t) \ \text{ if } \ x = (v,t) \ \text{ with } \ t \in (0\,,\,1)\,,$$

so that f' is well defined on $\mathbb{T} \setminus \mathbb{V}$. If f is edgewise C^1, we set

$$\begin{cases} f'(x) \;=\; f'_v(t) & \text{if } x \;=\; (v,t) \text{ with } t \in (0\,,\,1)\,, \\ f'_0(x) \;=\; f'_v(0) & \text{if } x \;=\; (v,0)\,, \\ f'_i(x) \;=\; f'_{v_i}(1) & \text{if } x \;=\; (v,0)\,, \end{cases}$$

where $\{v_i, 1 \le i \le \mathsf{r}\}$ is the set of all successors of v as introduced above. We will often abuse notation and write $f_i'(v)$ for $f_i'(v,0)$.

We say that a function f is edgewise twice differentiable (resp. edgewise C^2) if f is edgewise C^1 and f' is edgewise differentiable (resp. edgewise C^1). For such a function, we set $f'' = (f')'$.

We are now ready to introduce our "constant coefficients" diffusions on $\mathbb{T}$.

DEFINITION 4.2. Let α, β be two real parameters with $\beta > 0$. Let D_β be the space of all compactly supported, continuous edgewise C^2 functions satisfying the "bifurcation" condition

$$\beta \sum_1^{\mathsf{r}} f_i'(x) = f_0'(x) \quad \text{at each } x = (v, 0)\,. \tag{16}$$

On D_β, the "generator" $L = L_{\alpha,\beta}$ is defined by

$$Lf(x) = f''(x) + \alpha f'(x) \quad \text{if } x = (v,t)\,,\ t \in (0\,,\,1)\,. \tag{17}$$

LEMMA 4.3. *The operator $L = L_{\alpha,\beta}$ with domain D_β is symmetric in $L^2(\mathbb{T}, d\mu)$, where the density $d\mu$ of the measure $\mu = \mu_{\alpha,\beta}$ is given by*

$$d\mu(x) = m(x)\, d_{\mathbb{T}}x \quad \textit{with} \quad m(x) = m_{\alpha,\beta}(x) = \beta^{\mathfrak{n}(x)} e^{-\alpha \mathfrak{h}(x)}. \tag{18}$$

Proof. On each open edge $v \times (0\,,\,1)$, integration by parts gives

$$\begin{aligned}\int_0^1 f_v'(t) g_v'(t) \beta^{\mathfrak{n}(v)} e^{-\alpha\mathfrak{h}(v)+\alpha t}\, dt \;&=\; -\int_0^1 f(v,t)\big(g_v''(t) + \alpha g_v'(t)\big)\beta^{\mathfrak{n}(v)} e^{-\alpha\mathfrak{h}(v)+\alpha t}\, dt \\ &\quad +\; f_v(1) g_v'(1) \beta^{\mathfrak{n}(v)} e^{-\alpha\mathfrak{h}(v)+\alpha} - f_v(0) g_v'(0) \beta^{\mathfrak{n}(v)} e^{-\alpha\mathfrak{h}(v)}\,.\end{aligned}$$

Now, at a given vertex v, the residual term coming from the $\mathsf{r}+1$ incident edges is

$$\begin{aligned}&-\beta^{\mathfrak{n}(v)} e^{-\alpha\mathfrak{h}(v)} f_v(0) g_0'(v) + \beta^{\mathfrak{n}(v_i)} e^{-\alpha\mathfrak{h}(v_i)+\alpha} f_v(0) \sum_{i=1}^{\mathsf{r}} g_i'(v) \\ &\qquad = \beta^{\mathfrak{n}(v)} e^{-\alpha\mathfrak{h}(v)} f_v(0) \left(-g_0'(v) + \beta \sum_{i=1}^{\mathsf{r}} g_i'(v) \right),\end{aligned}$$

and it vanishes if g satisfies the boundary condition (17). □

In fact, $(L_{\alpha,\beta}, D_\beta)$ is essentially self-adjoint in $L^2(\mathbb{T}, d\mu_{\alpha,\beta})$. The proof requires some serious efforts, see Bendikov, Saloff-Coste, Salvatori and Woess [**2**], and compare also with Bendikov and Saloff-Coste [**1**]. The domain of its unique self-adjoint extension (still denoted $L = L_{\alpha,\beta}$) is the set of all functions in H^2_μ which are continuous and satisfy (16). This self-adjoint operator L is the infinitesimal generator of the Markov semigroup $H_\tau = e^{\tau L}$ associated with the Dirichlet form

$$D(f) = \int_{\mathbb{T}} |f'(x)|^2 d\mu(x)$$

with domain $\{f \in H^1_\mu : f \text{ continuous}\}$ in $L^2(\mathbb{T}, \mu)$. This Dirichlet form satisfies the doubling property and Poincaré inequality locally uniformly. By the very general results of Sturm [**21**], [**22**], [**23**], see also Saloff-Coste [**12**] and Eells and Fuglede [**6**], we have the following

PROPOSITION 4.4. *The semigroup H_τ admits a bounded continuous kernel $h^\mu_\tau(x,y)$ with respect to the reversible measure $\mu = \mu_{\alpha,\beta}$. That is,*

$$H_\tau f(x) = \int_{\mathbb{T}} h^\mu_\tau(x,y) f(y)\, d\mu(y)\,.$$

Moreover, for each integer $k \geq 0$,

$$\left|\partial^k_\tau\, h^\mu_\tau(x,y)\right| \leq C_k\, \tau^{-k}\, (\min\{1,\tau\})^{-1/2} \beta^{-\mathfrak{n}(x)} e^{\alpha \mathfrak{h}(x)} e^{-cd(x,y)^2/\tau} \tag{19}$$

for all $\tau > 0$ and $x, y \in \mathbb{T}$. As $h^\mu_\tau(x,y) = h^\mu_\tau(y,x)$, the factor $\beta^{-\mathfrak{n}(x)} e^{\alpha\mathfrak{h}(x)}$ can be replaced by $\beta^{-\mathfrak{n}(y)} e^{\alpha\mathfrak{h}(y)}$.

From this, it follows that H_τ can also be viewed as a semigroup acting on bounded continuous functions on $\mathbb{T}$ and that, in addition to the reversible measure $d\mu = m(\cdot)\, d_{\mathbb{T}}$, it admits also the measure $d_{\mathbb{T}}$ as invariant measure. As a matter of fact, for our computation, we will mostly use the kernel

$$h_\tau(x,y) = h^\mu_\tau(x,y) m(y)\,,$$

which is the kernel of H_τ with respect to the $\mathcal{G}$-invariant measure $d_{\mathbb{T}}$. Indeed, this kernel is invariant under the action of $\mathcal{G}$, i.e., for each $g \in \mathcal{G}$, $h_\tau(gx, gy) = h_\tau(x,y)$, whereas this is not true for $h^\mu_\tau(x,y)$. Note that (19) and the subsequent remark concerning symmetry give

$$\left|\partial^k_\tau\, h_\tau(x,y)\right| \leq C_k\, (\min\{1,\tau\})^{-1/2} e^{-c\, d(x,y)^2/\tau}\,. \tag{20}$$

We will also need similar estimates for the space derivatives, up to the bifurcation points, namely, for every closed interval $[a\,,\,b] \subset (0\,,\,\infty)$ and all integers $k, m \geq 0$, there is a constant $C_{a,b,k,m}$ such that

$$\sup_{\tau\in[a\,,\,b]}\ \sup_{s\in(0\,,\,1)} \left|\partial^k_\tau\, \partial^m_s\, h_\tau\big(x,(v,s)\big)\right| \leq C_{a,b,k,m}\, e^{-\bar{c}\, d(x,v)^2/b}\,. \tag{21}$$

Please note the space derivatives are not continuous through the bifurcation points. These non-trivial estimates are derived in [**2**] in a more general context.

4.C. Reduction to the factor space and computations. Applying Theorem 2.1, we can compute the norm and spectral radius of H_τ on various spaces. Recall that $I = \mathcal{G}\backslash\mathbb{T}$ is the circle parametrised by the interval $[0\,,\,1)$. For any $s \in [0\,,\,1)$, set $x_s = (o, s) \in \mathbb{T}$. We will need to use the action of some elements of G. Recall that o_i, $1 \leq i \leq \mathsf{r}$ are the "children" of o. For each $1 \leq i \leq \mathsf{r}$, let g_i be a fixed element of $\mathcal{G}$ such that $g_i o_i = o$. Since all elements of $\mathcal{G}$ fix ω, the image of o under g_i must be the unique predecessor of o.

Spectral radii on $L^p(\mathbb{T}, d\mu)$. Let $\mu = \mu_{\alpha,\beta}$ be the measure defined in Lemma 4.3. In the case of $L^p(\mathbb{T},\mu)$, Theorem 2.1 does not directly apply since μ is not invariant under G. Hence, we must instead use Remark 2.2 following Theorem 2.1. Accordingly, for a fixed $p \in (1,\infty)$, we consider the operator A_τ, $\tau > 0$, acting on $L^p(I, d\lambda)$ with kernel

$$\begin{aligned} a_\tau(s,t) &= \int_{X_t} \left(\frac{\gamma(y) m(x_s)}{\gamma(x_s) m(y)}\right)^{1/p} h_\tau(x_s, y)\, d_{X_t} y \\ &= \mathsf{r}^{-t} \sum_{v\in\mathbb{V}} (\mathsf{r}\, e^{-\alpha})^{-(\mathfrak{h}(v)-t+s)/p} \beta^{-\mathfrak{n}(v)/p}\, h_\tau\big(x_s, (v,t)\big). \end{aligned} \tag{22}$$

By (20) and (21), this is a bounded, smooth kernel on $[0\,,\,1)$. It follows that A_τ is a compact operator on $L^r(I, d\lambda)$ for any $1 < r < \infty$ (see, e.g., Schaefer [**17**, p.

283]) and, by (7), the family $(A_\tau)_{\tau>0}$ forms a semigroup of bounded operators. It follows from the Gaussian bound (20) that this semigroup is C_0. As any $L^r(I,d\lambda)$-eigenfunction for A_τ must be bounded, the spectrum of A_τ is independent of r. Let $\rho_\tau = \rho(A_\tau)$ be the spectral radius of A_τ on $L^r(I,d\lambda)$. Since $A_{\tau_1}A_{\tau_2} = A_{\tau_1+\tau_2}$, there are $\omega \ge 0$ and a positive function u on I such that

$$\rho_\tau = e^{-\omega\tau} \quad \text{and} \quad A_\tau u = e^{-\omega\tau} u \tag{23}$$

We are going to compute ω as the smallest real eigenvalue of a simple eigenvalue problem, see (25) below.

As a function of $s \in [0\,,\,1)$, $a_\tau(s,t)$ is smooth on $[0\,,\,1)$. We now study its one-sided limits at the endpoints. Recall that the interval parametrizes the circle. We shall see that $a_\tau(s,t)$ and its derivatives have jumps at the endpoints of the interval when the latter are identified on the circle. The following two limits exist:

$$\begin{aligned} a_\tau(1,t) = \lim_{s\to1} a_\tau(s,t) \quad &\text{and} \quad \partial_s^B a_\tau(1,t) = \lim_{s\to1} \partial_s^B a_\tau(s,t)\,, \\ \text{where} \quad B = (\mathsf{r}\, e^{-\alpha})^{-1/p} \quad &\text{and} \quad \partial_s^B = B^s \partial_s B^{-s}\,. \end{aligned} \tag{24}$$

Lemma 4.5. (a) *We have*

$$a_\tau(1,t) = \beta^{1/p}\, a_\tau(0,t) \quad \textit{and} \quad \beta\, \mathsf{r}\, \partial_s^B a_\tau(1,t) = \beta^{1/p}\, \partial_s^B\, a_\tau(0,t)\,.$$

(b) *For* $s \in (0\,,\,1)$,

$$\partial_\tau a_\tau(s,t) = \big(\partial_s^B\big)^2 a_\tau(s,t) + \alpha\, \partial_s^B a_\tau(s,t)\,.$$

Proof. Regarding the first identity of (a), recall that $g_1 o$ is the unique predecessor of o. We compute

$$\begin{aligned} \mathsf{r}^t a_\tau(1,t) &= \sum_{v\in\mathbb{V}} B^{\mathfrak{h}(v)-t+1}\, \beta^{-\mathfrak{n}(v)/p}\, h_\tau\big((g_1 o),(v,t)\big) \\ &= \sum_{v\in\mathbb{V}} B^{\mathfrak{h}(v)-t+1}\, \beta^{-\mathfrak{n}(v)/p}\, h_\tau\big(o,(g_1^{-1}v,t)\big) \\ &= \sum_{v\in\mathbb{V}} B^{\mathfrak{h}(g_1^{-1}v)-t}\, \beta^{-(\mathfrak{n}(g_1^{-1}v)-1)/p}\, h_\tau\big(o,(g_1^{-1}v,t)\big) \\ &= \sum_{v\in\mathbb{V}} B^{\mathfrak{h}(v)-t}\, \beta^{-(\mathfrak{n}(v)-1)/p}\, h_\tau\big(o,(v,t)\big) = \beta^{1/p}\, \mathsf{r}^t\, a_\tau(0,t) \end{aligned}$$

For the second identity of (a), set $h_\tau^y(x) = h_\tau(x,y)$. Then

$$\mathsf{r}^t\, \partial_s^B a_\tau(s,t) = \sum_{v\in\mathbb{V}} B^{\mathfrak{h}(v)-t+s}\, \beta^{-\mathfrak{n}(v)/p}\, \partial_s h_\tau^{(v,t)}(x_s).$$

Observe that for each $i \in \{1,\dots,\mathsf{r}\}$, we have

$$\begin{aligned} \lim_{s\to1} \partial_s h_\tau^y\big((o,s)\big) &= \lim_{s\to1} \partial_s h_\tau^y\big((g_i o_i,s)\big) \\ &= \lim_{s\to1} \partial_s h_\tau^{g_i^{-1}y}\big((o_i,s)\big) = \big(h_\tau^{g_i^{-1}y}\big)_i'(o), \end{aligned}$$

and compute

$$\begin{aligned}\mathsf{r}^t\,\partial_s^B a_\tau(1,t) &= \sum_{v\in\mathbb{V}} B^{\mathfrak{h}(v)-t+1}\,\beta^{-\mathfrak{n}(v)/p}\,\big(h_\tau^{(g_i^{-1}v,t)}\big)'_i(o)\\ &= \sum_{v\in\mathbb{V}} B^{\mathfrak{h}(g_i^{-1}v)-t}\,\beta^{-(\mathfrak{n}(g_i^{-1}v)-1)/p}\,\big(h_\tau^{(g_i^{-1}v,t)}\big)'_i(o)\\ &= \beta^{1/p}\sum_{v\in\mathbb{V}} B^{\mathfrak{h}(v)-t}\,\beta^{-\mathfrak{n}(v)/p}\,\big(h_\tau^{(v,t)}\big)'_i(o)\,.\end{aligned}$$

Now the identity follows after recalling that the function h_τ^y must satisfy (16), that is,

$$\beta\sum_1^{\mathsf{r}}\big(h_\tau^y\big)'_i(v) = \big(h_\tau^y\big)'_0(v).$$

Finally, for $s\in(0\,,\,1)$, $a_\tau(s,t)$ is smooth, and by (17),

$$\partial_\tau h_\tau\big((o,s),y\big) = \partial_s^2 h_\tau\big((o,s),y\big) + \alpha\partial_s h_\tau\big((o,s),y\big)\,.$$

The identity (b) follows. □

(23) and Lemma 4.5 lead to the eigenvalue problem

$$\begin{cases}\big(\partial_s^B\big)^2u(s)+\alpha\,\partial_s^Bu(s) = -\omega\,u(s)\quad\text{on }\ (0\,,\,1)\\ u(1)=\beta^{1/p}\,u(0)\\ \mathsf{r}\,\beta\,\partial_s^Bu(1)=\beta^{1/p}\,\partial_s^Bu(0)\end{cases}\tag{25}$$

where the values of u and $\partial_s^B u$ at 0 and 1 must be understood as limits as s tends to either 0 or 1. Setting $v(s)=B^{-s}u(s)$, we find the following.

Corollary 4.6. *The spectral radius of A_τ on $L^p(I,\lambda)$ is $\rho_\tau=e^{-\omega\tau}$, where ω is the smallest real solution of*

$$\begin{cases}v''(s)+\alpha\,v'(s) = -\omega\,v(s)\quad\textit{on }\ (0\,,\,1)\\ v(1)=b^{1/p}\,v(0)\,,\\ v'(1)=(\mathsf{r}\,\beta)^{-1}\,a^{1/p}\,v'(0)\,,\end{cases}\tag{26}$$

where $b=b(\alpha,\beta)=B^{-p}\,\beta=\mathsf{r}\,\beta\,e^{-\alpha}>0\,.$

We can now solve (4.6). If $4\omega\neq\alpha^2$ and $\omega>0$, the general real solution of $v''+\alpha\,v'+\omega\,v=0$ is

$$v(s)=c_1\,e^{(-\alpha+z)s/2}+c_2\,e^{(-\alpha-z)s/2}$$

with

$$z=\begin{cases}\sqrt{\alpha^2-4\omega}\,, & \text{if }\ \alpha^2>4\omega\\ \mathrm{i}\sqrt{4\omega-\alpha^2}\,, & \text{if }\ \alpha^2<4\omega\,.\end{cases}$$

We are lead to the linear system

$$\begin{cases}c_1\left(e^{(-\alpha+z)/2}-b^{1/p}\right)+c_2\left(e^{(-\alpha-z)/2}-b^{1/p}\right) &= 0\\ c_1(-\alpha+z)\left(e^{(-\alpha+z)/2}-(\mathsf{r}\,\beta)^{-1}\,b^{1/p}\right)+c_2(-\alpha-z)\left(e^{(-\alpha-z)/2}-(\mathsf{r}\,\beta)^{-1}\,b^{1/p}\right) &= 0\,.\end{cases}$$

This linear system has a non-trivial solution (c_1,c_2) if and only if its determinant is 0, which leads to the equation

$$\alpha(\beta\,\mathsf{r}-1)\frac{\sinh(z/2)}{z} = (\beta\,\mathsf{r}+1)\cosh(z/2)-e^{\alpha/2}\big((\mathsf{r}\,\beta\,e^{-\alpha})b^{-1/p}+b^{1/p}\big).\tag{27}$$

If we substitute $z = \mathsf{i}\,\zeta$ then (27) becomes

$$\alpha(\beta\,\mathsf{r}-1)\frac{\sin(\zeta/2)}{\zeta} = (\beta\,\mathsf{r}+1)\cos(\zeta/2) - e^{\alpha/2}\big((\mathsf{r}\,\beta\,e^{-\alpha})b^{-1/p} + b^{1/p}\big). \tag{28}$$

The last term can of course be simplified to

$$\big((\mathsf{r}\,\beta\,e^{-\alpha})b^{-1/p} + b^{1/p}\big) = (b^{1/p} + b^{1-1/p})\,,$$

but we shall need it in the above form later on. We can now summarize the computation of ω. In the following theorem, we first consider three special cases.

THEOREM 4.7. *The spectral gap $\omega = \omega_p(\alpha,\beta)$ of the operator $L_{\alpha,\beta}$ acting on the space $L^p(\mathbb{T}, d\mu_{\alpha,\beta})$ is given as follows.*

(I) No continuous drift: $\alpha = 0$. *Then*

$$\omega = \left[\arccos\left(\frac{(\mathsf{r}\beta)^{1/p} + (\mathsf{r}\beta)^{1-1/p}}{1+\mathsf{r}\beta}\right)\right]^2.$$

In particular, if $\alpha = 0$ and $\beta = 1/\mathsf{r}$ then $\omega = 0$.

(II) No discrete drift: $\beta = 1/\mathsf{r}$. *Then*

$$\omega = \alpha^2 p^{-1}\left(1 - p^{-1}\right).$$

(III) No total drift: $b(\alpha,\beta) = \mathsf{r}\,\beta\,e^{-\alpha} = 1$. *Then $\omega = 0$, there is no spectral gap.*

(IV) The general case: $\alpha \neq 0$, $\beta\mathsf{r} \neq 1$, $\mathsf{r}\,\beta\,e^{-\alpha} \neq 1$.

(IV.i) *If* $\alpha(\beta\,\mathsf{r}-1) < 2\Big(\beta\,\mathsf{r} + 1 - e^{\alpha/2}\big(b(\alpha,\beta)^{1/p} + b(\alpha,\beta)^{1-1/p}\big)\Big)$ *then*

$$\omega = \frac{1}{4}(\alpha^2 + \zeta_0^2)\,,$$

where ζ_0 is the smallest positive solution (in ζ) of (28).

(IV.ii) *If* $\alpha(\beta\,\mathsf{r}-1) \geq 2\Big(\beta\,\mathsf{r} + 1 - e^{\alpha/2}\big(a(\alpha,\beta)^{1/p} + a(\alpha,\beta)^{1-1/p}\big)\Big)$ *then*

$$\omega = \frac{1}{4}(\alpha^2 - z_0^2)\,,$$

where z_0 is the unique positive real solution of (27).

Proof. (I) When $\alpha = 0$, equation (28) becomes

$$\cos\sqrt{\omega} = \frac{(\mathsf{r}\,\beta)^{1/p} + (\mathsf{r}\,\beta)^{1-1/p}}{1+\mathsf{r}\,\beta}.$$

Note that the right hand side is in $(0\,,\,1]$. In the special case $\alpha = 0\,,\ \beta = 1/\mathsf{r}$, we have no spectral gap ($\omega = 0$ is solution) and (26) with $\omega = 0$ admits the constant function $v \equiv 1$ as a solution.

(II) When $\beta = 1/\mathsf{r}$, equation (27) gives

$$\cosh(z/2) = \cosh\big(\alpha(1/2 - 1/p)\big),$$

leading to the proposed smallest positive solution for ω.

(III) If $b(\alpha,\beta) = 1$, it is easy to check directly that (26) admits $\omega = 0$, $v \equiv 1$ has a solution.

(IV.i) In this case, (27) has no real solution, so that we need the smallest positive real solution $\zeta_0 = \zeta_0(\mathsf{r},\alpha,\beta,p)$ of (28).

(IV.ii) In the second case, (27) has a unique real positive solution $z_0 = z(\mathsf{r}, \alpha, \beta, p)$, and $\omega = (\alpha^2 - z_0^2)/4$.

In particular, when $\alpha(\beta\,\mathsf{r} - 1) = 2\Big(\beta\,\mathsf{r} + 1 - e^{\alpha/2}\big(b(\alpha,\beta)^{1/p} + b(\alpha,\beta)^{1-1/p}\big)\Big)$ then the characteristic equation of the differential equation $v'' + \alpha v' + \omega v = 0$ has a double root, and the general form of the solution is $(c_1 + c_2\, s)e^{-\alpha s/2}$. Thus, one easily checks that $\omega = \alpha^2/4$ is indeed a solution. □

We remark that in the case $p = 2$, our analysis shows that

$$\omega \ge \alpha^2/4 \quad \text{if and only if} \quad \frac{\alpha(1+\sqrt{\mathsf{r}\,\beta})}{\sqrt{\mathsf{r}\,\beta}-1} \le 1\,.$$

We also remark that Cattaneo [**4**] considers an analog of the operator $L_{0,\beta}$ on general networks and relates the L^2-spectrum of that Laplacian with the ℓ^2-spectrum of the transition operator on the vertex set of the network. In particular, on can combine her results with the well-known formula for the spectral radius of simple random walk on the tree to obtain $\omega = \omega_2(0\,,\,1)$, corresponding to case (I) of Theorem 4.7 with $\beta = 1$ and $p = 2$.

Spectral radii with respect to other measures on $\mathbb{T}$*.* By (20), the semigroup $H_\tau = e^{-tL}$ not only acts on $L^p(\mathbb{T}, d\mu_{\alpha,\beta})$, but also on $L^p(\mathbb{T}, d\mu_{\kappa,\eta})$, with the measure $\mu_{\kappa,\eta}$ defined as in (18) for any $\kappa \in \mathbb{R}$ and $\eta > 0$ in the place of α and β. In particular, $L^p(\mathbb{T}, d_{\mathbb{T}})$ corresponds to $\kappa = 0\,,\ \eta = 1$.

Accordingly, for a fixed $p \in (1,\infty)$, we consider the operator A_τ, $\tau > 0$, acting on $L^p(I, d\lambda)$ with kernel

$$\begin{aligned} a_\tau(s,t) &= \int_{X_t} \left(\frac{\gamma(y) m_{\kappa,\eta}(x_s)}{\gamma(x_s) m_{\kappa,\eta}(y)}\right)^{1/p} h_\tau(x_s, y)\, d_{X_t} y \\ &= \mathsf{r}^{-t} \sum_{v\in\mathbb{V}} (\mathsf{r}\, e^{-\kappa})^{-(\mathfrak{h}(v)-t+s)/p}\, \eta^{-\mathfrak{n}(v)}\, h_\tau\big(x_s, (v,t)\big)\,. \end{aligned}$$

An analysis similar to that of the special case of $L^p(\mathbb{T}, d\mu_{\alpha,\beta})$ shows that the spectral radius of A_τ is $e^{-\omega\tau}$ where ω is the smallest real solution of the same eigenvalue problem as in (26), but with different constant

$$b = b(\kappa,\eta) = \mathsf{r}\,\eta\, e^{-\kappa} > 0\,. \tag{29}$$

In this situation, since $L_{\alpha,\beta}$ is in general not negative semi-definite on $L^p(\mathbb{T}, d\mu_{\kappa,\eta})$, we can no more speak of the "spectral gap", since ω may also become negative. We get the following results (with arcosh $= \cosh^{-1}$).

THEOREM 4.8. *With $b = b(\kappa,\eta)$ as in (29), the top of the spectrum $-\omega = -\omega_p(\alpha,\beta;\kappa,\eta)$ of the operator $L_{\alpha,\beta}$ acting on the space $L^p(\mathbb{T}, d\mu_{\kappa,\eta})$ is given as follows.*

(I) No continuous drift: $\alpha = 0$. *Then*

$$\omega = \begin{cases} \left[\arccos\left(\dfrac{b^{1/p} + \mathsf{r}\,\beta\, b^{-1/p}}{1+\mathsf{r}\,\beta}\right)\right]^2, & \text{if } (b^{1/p} - \mathsf{r}\,\eta)(b^{1/p}-1) < 0, \\ \left[\operatorname{arcosh}\left(\dfrac{b^{1/p} + \mathsf{r}\,\beta\, b^{-1/p}}{1+\mathsf{r}\,\beta}\right)\right]^2, & \text{if } (b^{1/p} - \mathsf{r}\,\eta)(b^{1/p}-1) \ge 0\,. \end{cases}$$

(II) No discrete drift: $\beta = 1/\mathsf{r}$. *Then*

$$\omega = \alpha \frac{\log b}{p} - \left| \frac{\log b}{p} \right|^2 .$$

(III) The case $\mathsf{r}\,\eta\, e^{-\kappa} = 1$: *Here,* $b = 1$ *and* $\omega = 0$.

(IV) The general case: $\alpha \neq 0$, $\mathsf{r}\,\beta \neq 1$, $\mathsf{r}\,\eta\, e^{-\kappa} \neq 1$.

(IV.i) *If* $\alpha(\mathsf{r}\,\beta - 1) < 2\Big((\mathsf{r}\,\beta + 1) - e^{\alpha/2}\big((\mathsf{r}\,\beta\, e^{-\alpha})\, b^{-1/p} + b^{1/p}\big)\Big)$ *then*

$$\omega = \frac{1}{4}(\alpha^2 + \zeta_0^2)\,,$$

where ζ_0 *is the smallest positive solution (in* ζ*) of* (28).

(IV.ii) *If* $\alpha(\beta\,\mathsf{r} - 1) \geq 2\Big((\mathsf{r}\,\beta + 1) - e^{\alpha/2}\big((\mathsf{r}\,\beta\, e^{-\alpha})\, b^{-1/p} + b^{1/p}\big)\Big)$ *then*

$$\omega = \frac{1}{4}(\alpha^2 - z_0^2)\,,$$

where z_0 *is the unique positive real solution of* (27).

5. Convolution on locally compact connected groups

5.A. Unravelling Theorem 2.1. This section relates Theorem 2.1 to questions of classic harmonic analysis. To preserve natural notation, we will have to change our own notation a bit. Let $X = \mathcal{G}$ be a locally compact group equipped with a fixed left Haar measure $d_{\mathcal{G}}$. Denote by $\Delta_{\mathcal{G}}$ the modular function on $\mathcal{G}$.

While primarily we have in mind the case when the group is connected, we shall see on several occasions that this is not necessarily required.

Let us assume that the kernel $k(x,y) \geq 0$ is a continuous function of (x,y), that $\int_X k(x,y) d_{\mathcal{G}} y = 1$, and that the operator

$$Kf(x) = \int_X k(x,y) f(y)\, d_{\mathcal{G}} y \tag{30}$$

is invariant under the left action of $\mathcal{G}$. In this case, if we set

$$\phi(y) = \phi_{\mathcal{G}}(y) = k(e, y^{-1}) = k(y,e), \quad d\Phi = \phi\, d_{\mathcal{G}}\,, \tag{31}$$

we have

$$\begin{aligned} Kf(x) = f * \Phi(x) &= \int_{\mathcal{G}} \phi(y^{-1}x)\, f(y)\, d_{\mathcal{G}} y \\ &= \int_{\mathcal{G}} f(xy^{-1})\, \Delta_{\mathcal{G}}(y)^{-1}\, \phi(y)\, d_{\mathcal{G}}\, y \end{aligned} \tag{32}$$

that is, $K = R_\Phi$ is the operator of right convolution by the probability measure Φ (or equivalently, by its density, the function ϕ) on $\mathcal{G}$. Obviously, we could as well have started with ϕ on $\mathcal{G}$ and set $k(x,y) = \phi(y^{-1}x)$. We are interested in computing the norm $\sigma_p(K) \in (0\,,\,1]$ and spectral radius $\rho_p(K) \in (0\,,\,1]$ of $K = R_\Phi$ acting on $L^p(\mathcal{G}, d_{\mathcal{G}})$, $1 < p < \infty$ (*right* convolution operator, but *left* Haar measure !).

By Theorem 2.1, these numbers equal one if and only if $\mathcal{G}$ is both amenable and unimodular. Our goal is to get a better understanding of what happens when $\mathcal{G}$ is non-unimodular or non-amenable or both, under the additional assumption that $\mathcal{G}$ is connected. It will be useful to proceed in several steps.

Step 1: reduction to the semisimple case with finite center. We start with the following simple application of Theorem 2.1, valid also without assuming connectedness. For connected groups it provides the reduction addressed above.

PROPOSITION 5.1. *Let $\mathcal{G}$ be a locally compact group and let K, k, ϕ be as above. Let $\mathcal{Q}$ be a closed normal amenable subgroup and $\mathcal{L} = \mathcal{Q}\backslash\mathcal{G}$. Then, for any $1 \leq p \leq \infty$,*

$$\sigma_p(K) = \sigma_p(K_{\mathcal{L}}) \quad \text{and} \quad \rho_p(K) = \rho_p(K_{\mathcal{L}}),$$

where $K_{\mathcal{L}}$ is the left invariant operator acting on $L^p(\mathcal{L})$ associated with the kernel

$$k_{\mathcal{L}}(u,v) = \left(\frac{\Delta_{\mathcal{G}}(x_v^{-1}x_u)}{\Delta_{\mathcal{L}}(v^{-1}u)}\right)^{1/p} \frac{\Delta_{\mathcal{G}}(x_v)}{\Delta_{\mathcal{L}}(v)} \int_{\mathcal{Q}} \Delta_{\mathcal{G}}(y)^{-1/p}\, k(x_u, yx_v)\; d_{\mathcal{Q}}y.$$

Here $x_u \in \mathcal{G}$ denotes any representative of $u \in \mathcal{L}$.

Proof. By Bourbaki [**3**, chap. VII, §2], we have $\Delta_{\mathcal{G}}(x) = \Delta_{\mathcal{Q}}(x)$ for all $x \in \mathcal{Q}$. Moreover, there exists a continuous decomposition of $d_{\mathcal{G}}x$ under the left action of $\mathcal{Q}$ such that the mesure $d\lambda$ on the quotient $\mathcal{L} = I = \mathcal{Q}\backslash\mathcal{G}$ is the left Haar measure on $\mathcal{L}$. Note that in [**3**, chap VII, §2], the quotient by $\mathcal{Q}$ (called G' in [**3**]) is the right quotient so that some translation is needed to obtain the desired decomposition. Namely, for any $f \in \mathcal{C}_{00}(X)$ we have

$$\int_{\mathcal{G}} f(g)\, d_{\mathcal{G}}g = \int_{\mathcal{L}} \left\{ \frac{\Delta_{\mathcal{G}}(x_u)}{\Delta_{\mathcal{L}}(u)} \int_{\mathcal{Q}} f(yx_u)\, d_{\mathcal{Q}}y \right\} d_{\mathcal{L}}u\,.$$

In this decomposition, the invariant measure $d_{X_u}\xi$ on the orbit

$$X_u = \{\xi = yx_u : y \in \mathcal{Q}\}$$

equals

$$d_{X_u}\xi = \frac{\Delta_{\mathcal{G}}(x_u)}{\Delta_{\mathcal{L}}(u)}\, d_{\mathcal{Q}}y\,.$$

Note that here $\{e\} = \mathcal{G}_x \subset \mathcal{Q} \subset \mathcal{G}$ is the stabilizer of $x \in \mathcal{G}$ under the left action of $\mathcal{Q}$. Hence, formula (1) shows that, for each $u \in \mathcal{L}$, $x \in X_u$, we have

$$\gamma(x) = |\{e\}|_{\mathcal{G}_x} = \frac{\Delta_{\mathcal{L}}(u)}{\Delta_{\mathcal{G}}(x)}\,.$$

With these remarks, the proposition is immediately deduced from Theorem 2.1. □

REMARK 5.2. By [**3**, chap. VII, §2], if for $x \in \mathcal{G}$ we set $\phi_x : \mathcal{Q} \to \mathcal{Q}, y \mapsto x^{-1}yx$ then

$$\int_{\mathcal{Q}} f\left(\phi_{x_u}^{-1}(y)\right) d_{\mathcal{Q}}y = \frac{\Delta_{\mathcal{G}}(x_u)}{\Delta_{\mathcal{L}}(u)} \int f(y)\, d_{\mathcal{Q}}y\,.$$

It follows that we can write

$$\begin{aligned} k_{\mathcal{L}}(u,v) &= \left(\frac{\Delta_{\mathcal{G}}(x_v^{-1}x_u)}{\Delta_{\mathcal{L}}(v^{-1}u)}\right)^{1/p} \int_{\mathcal{Q}} \Delta_{\mathcal{G}}(y)^{-1/p}\, k(x_u, x_v y)\, d_{\mathcal{Q}}y \\ &= \left(\frac{\Delta_{\mathcal{G}}(x_v^{-1}x_u)}{\Delta_{\mathcal{L}}(v^{-1}u)}\right)^{1/p} \int_{\mathcal{Q}} \Delta_{\mathcal{G}}(y)^{-1/p}\, k(x_v^{-1}x_u, y)\, d_{\mathcal{Q}}y\,. \end{aligned}$$

This shows that the operator $K_{\mathcal{L}}$ is a (right) convolution operator on $\mathcal{L}$, i.e.,

$$k_{\mathcal{L}}(u,v) = \phi_{\mathcal{L}}(v^{-1}u)$$

depends only on $v^{-1}u$. The correspondence between ϕ and $\phi_{\mathcal{L}}$ is given by

$$\phi_{\mathcal{L}}(u) = \Delta_{\mathcal{L}}(u)^{-1/p} \int_{\mathcal{Q}} \Delta_{\mathcal{G}}(y^{-1}x_u)^{1/p}\, \phi(y^{-1}x_u)\, d_{\mathcal{Q}}y\,.$$

For $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$, denote by $\phi_{p,\mathcal{L}}, \phi_{q,\mathcal{L}}$ the corresponding functions on $\mathcal{L}$. Then, we have

$$\begin{aligned}(\check{\phi})_{p,\mathcal{L}}(u) &= \Delta_{\mathcal{L}}(u)^{-1/p} \int_{\mathcal{Q}} \Delta_{\mathcal{G}}(x_{u^{-1}}y)^{-1/p}\, \phi(x_{u^{-1}}y)\, d_{\mathcal{Q}}y \\ &= \Delta_{\mathcal{L}}(u)^{1/q} \int_{\mathcal{Q}} \Delta_{\mathcal{G}}(yx_{u^{-1}})^{-1/p} \Delta_{\mathcal{G}}(x_{u^{-1}})\, \phi(yx_{u^{-1}})\, d_{\mathcal{Q}}y \\ &= \Delta_{\mathcal{L}}(u^{-1})^{-1/q} \int_{\mathcal{Q}} \Delta_{\mathcal{G}}(y^{-1}x_{u^{-1}})^{1/q}\, \phi(y^{-1}x_{u^{-1}})\, d_{\mathcal{Q}}y \\ &= \phi_{q,\mathcal{L}}(u^{-1}) = (\phi_{q,\mathcal{L}})\check{}\,(u)\,.\end{aligned}$$

This is consistent with the fact that if K is the operator of right convolution by a function $\check{\phi}$ on a group $\mathcal{G}$, its formal adjoint K^* is right convolution by ϕ and for $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, $\sigma_q(K) = \sigma_p(K^*)$.

Step 2: unimodular groups of type $\mathcal{PK}$, where $\mathcal{P}$ is amenable and $\mathcal{K}$ compact. Let $\mathcal{L}$ be a locally compact unimodular group such that there are closed subgroups $\mathcal{P}, \mathcal{K}$ with $\mathcal{P}$ amenable and $\mathcal{K}$ compact such that $\mathcal{L} = \mathcal{PK}$. Let

$$k_{\mathcal{L}}(u, v) = \phi_{\mathcal{L}}(v^{-1}u)$$

be a left invariant kernel under the action of $\mathcal{L}$. In particular, $k_{\mathcal{L}}$ is left invariant under the action of the closed amenable subgroup $\mathcal{P}$. Thus we can apply Theorem 2.1. In general, $\mathcal{P}$ is not normal in $\mathcal{L}$ and the quotient space

$$I = \mathcal{P}\backslash\mathcal{L} = [\mathcal{K} \cap \mathcal{P}]\backslash\mathcal{K}$$

is compact. Let $d\lambda$ be the unique K invariant measure on I such that the normalized Haar measures on $\mathcal{K}$ and $\mathcal{K} \cap \mathcal{P}$ satisfy $d_{\mathcal{K}}v = d_{\mathcal{K}\cap\mathcal{P}}x d\lambda(s)$. Classical results show that there is a continuous decomposition of the Haar measure on $\mathcal{L}$ as

$$d_{\mathcal{L}}u = d_{\mathcal{P}}x\, d\lambda(s)\,, \quad s \in I,\ u = s \mod \mathcal{P}$$

with $\gamma(x) = \Delta_{\mathcal{P}}(x)^{-1}$. Again, $\gamma(x) = |\{e\}|_{\mathcal{G}_x}$ where $\{e\} = \mathcal{G}_x \subset \mathcal{P} \subset \mathcal{L}$ is viewed as the stabilizer of $x \in \mathcal{L}$ under the left action of $\mathcal{P}$.

For any $u = xv \in \mathcal{L}$ with $x \in \mathcal{P}$, $v \in \mathcal{K}$, we set

$$\Delta^{\mathcal{L}}_{\mathcal{P}}(u) = \Delta_{\mathcal{P}}(x). \tag{33}$$

This makes sense because if $u = xv \in \mathcal{L}$, $x \in \mathcal{P}$, $v \in \mathcal{K}$ and $u = x'v' \in \mathcal{L}$ with $x' \in \mathcal{P}$, $v' \in \mathcal{K}$ then $v'v^{-1} \in \mathcal{P} \cap \mathcal{K}$ and thus $\Delta_{\mathcal{P}}(v'v^{-1}) = 1$ ($\mathcal{P} \cap \mathcal{K}$ is a compact). Hence $\Delta_{\mathcal{P}}(x') = \Delta_{\mathcal{P}}(x)$. We insist in using this somewhat cumbersome notation instead of a lighter alternative such as $\Delta_{\mathcal{P}}(u)$, because in general, for $u, v \in \mathcal{L}$,

$$\Delta^{\mathcal{L}}_{\mathcal{P}}(uv) \neq \Delta^{\mathcal{L}}_{\mathcal{P}}(u)\, \Delta^{\mathcal{L}}_{\mathcal{P}}(v).$$

In this context, Theorem 2.1 leads to the study of the operator with kernel

$$k_I(s, t) = \int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p} k_{\mathcal{L}}(s, xt)\, d_{\mathcal{P}}x\,.$$

acting on $L^p(I, d\lambda)$, $1 < p < \infty$. However, one can lift this operator from the quotient space $I = [\mathcal{K} \cap \mathcal{P}]\backslash\mathcal{K}$ to $\mathcal{K}$ in an obvious way without changing its norm or spectral radius. This gives the following.

PROPOSITION 5.3. *Let $\mathcal{L}$ be a unimodular locally compact group such that $\mathcal{L} = \mathcal{P}\mathcal{K}$ for two closed subgroups $\mathcal{P}, \mathcal{K}$ with $\mathcal{P}$ amenable and $\mathcal{K}$ compact. Let $k_{\mathcal{L}}(u,v) = \phi_{\mathcal{L}}(v^{-1}u)$ be a left invariant kernel on $\mathcal{L}$ as above. Fix $p, q \in [1,\infty]$ with $1/p + 1/q = 1$. Then*

$$\sigma_p(K_{\mathcal{L}}) = \sigma_p(K_{\mathcal{K}}) \quad \textit{and} \quad \rho_p(H_{\mathcal{L}}) = \rho_p(K_{\mathcal{K}})\,,$$

where $K_{\mathcal{K}}$ is the operator on $L^p(\mathcal{K})$ with kernel

$$k_{\mathcal{K}}(s,t) = \int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p}\, k_{\mathcal{L}}(s, xt)\; d_{\mathcal{P}}x\,.$$

In particular, we have the following results:

(a)

$$\sigma_p(K_{\mathcal{L}}) \le \left\{ \int_{\mathcal{K}} \left[\int_{\mathcal{K}} \left(\int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p}\, k_{\mathcal{L}}(s,xt)\, d_{\mathcal{P}}x \right)^q d_{\mathcal{K}}t \right]^{p/q} d_{\mathcal{K}}s \right\}^{1/p} .$$

(b) *Assume that $k_{\mathcal{L}}(e, vs) = k_{\mathcal{L}}(e, sv)$ for all $s \in \mathcal{K}$, $v \in \mathcal{L}$. Then*

$$\sigma_p(\mathcal{K}_{\mathcal{L}}) = \rho_p(K_{\mathcal{L}}) = \int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/p}\, k_{\mathcal{L}}(e,v)\, d_{\mathcal{L}}v\,.$$

(c) *Assume that $k_{\mathcal{L}}(e, vs) = k_{\mathcal{L}}(e, v)$ for all $s \in \mathcal{K}, v \in \mathcal{L}$. Then*

$$\sigma_p(K_{\mathcal{L}}) = \left[\int_{\mathcal{K}} \left(\int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p} k_{\mathcal{L}}(s,x) d_{\mathcal{P}}x \right)^p d_{\mathcal{K}}s \right]^{1/p}$$

and, assuming $\sigma_p(K_{\mathcal{L}}) < \infty$,

$$\rho_p(K_{\mathcal{L}}) = \int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/q}\, k_{\mathcal{L}}(v,e)\, d_{\mathcal{L}}v\,.$$

(d) *Assume that $k_{\mathcal{L}}(e, sv) = k_{\mathcal{L}}(e, v)$ for all $s \in \mathcal{K}, v \in \mathcal{L}$. Then*

$$\sigma_p(K_{\mathcal{L}}) = \left[\int_{\mathcal{K}} \left(\int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p}\, k_{\mathcal{L}}(e,xs)\, d_{\mathcal{P}}x \right)^q d_K s \right]^{1/q}$$

and, assuming $\sigma_p(K_{\mathcal{L}}) < \infty$,

$$\rho_p(K_{\mathcal{L}}) = \int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/p}\, k_{\mathcal{L}}(e,v)\, d_{\mathcal{L}}v\,.$$

Proof. To prove this proposition, we apply Theorem 2.1. Part (a) follows from a standard estimation of $\sigma_p(K_{\mathcal{K}})$.

For (b), observe that $k_{\mathcal{K}}(s,t)$ depends only on $t^{-1}s$. Hence $K_{\mathcal{K}}$ is a convolution operator with non-negative kernel on $\mathcal{K}$ and its norm equals its spectral radius, which equals $\int_{\mathcal{K}} k_{\mathcal{K}}(e,s)\, d_{\mathcal{K}}s$.

We have $K_{\mathcal{K}} f(s) = \big(\int_{\mathcal{K}} f \, d_{\mathcal{K}}\big)\, k_{\mathcal{K}}(s,e)$. The formula for $\sigma_p(K_{\mathcal{L}})$ follows. Moreover,

$$
\begin{aligned}
\rho_p(\mathcal{K}_{\mathcal{L}}) &= \int_{\mathcal{K}} \int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p}\, k_{\mathcal{L}}(s,x)\, d_{\mathcal{P}}x\, d_{\mathcal{K}}s \\
&= \int_{\mathcal{K}} \int_{\mathcal{P}} \Delta_{\mathcal{P}}(x^{-1})^{-1/q}\, \Delta_{\mathcal{P}}(x^{-1})\, k_{\mathcal{L}}(x^{-1}s,e)\, d_{\mathcal{P}}x\, d_{K}s \\
&= \int_{\mathcal{K}} \int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/q}\, k_{\mathcal{L}}(xs,e)\, d_{\mathcal{P}}x\, d_{\mathcal{K}}s \\
&= \int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/q}\, k_{\mathcal{L}}(v,e)\, d_{\mathcal{L}}v\,,
\end{aligned}
\tag{34}
$$

as stated.

In (d) we have $K_{\mathcal{K}} f(s) = \int_{\mathcal{K}} k_{\mathcal{K}}(e,t) f(t)\, d_{\mathcal{K}}t$, and the desired results follow since

$$
\rho_p(K_{\mathcal{L}}) = \int_{\mathcal{K}} \int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p}\, k_{\mathcal{L}}(e,xs)\, d_{\mathcal{P}}x\, d_{\mathcal{K}}s = \int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/p}\, k_{\mathcal{L}}(e,v)\, d_{\mathcal{L}}v\,.
$$

Note that (c) and (d) are dual to each other by passing to the adjoint operator (i.e., changing $k_{\mathcal{L}}(u,v)$ to $k_{\mathcal{L}}(v,u)$). □

The following statement is a simple corollary of Proposition 5.3 together with the characterization of amenability in terms of convolution operator norms (see [**5**, Prop. 4.2] and [**15**]).

Proposition 5.4. *Let $\mathcal{L}$ be a locally compact unimodular group of type $\mathcal{PK}$ with $\mathcal{P},\mathcal{K}$ two closed subgroups, $\mathcal{P}$ amenable, $\mathcal{K}$ compact. Then $\mathcal{L}$ is non-amenable if and only if $\mathcal{P}$ is non-unimodular.*

The results of this section apply to real connected semisimple Lie group with finite center and to the group of $\mathfrak{K}$-points of a connected (in the Zariski topology) linear algebraic semisimple group defined over the local field $\mathfrak{K}$. In both cases there are Iwasawa decompositions that yield $\mathcal{L} = \mathcal{PK}$ with $\mathcal{P},\mathcal{K}$ closed subgroups, $\mathcal{P}$ amenable, $\mathcal{K}$ compact. In the general case of algebraic groups, the intersection of $\mathcal{P}$ and $\mathcal{K}$ may not be reduced to $\{e\}$. Other interesting cases arise as subgroups of the automorphism group of a free group, see Nebbia [**10**]. We next give details in the case of connected Lie groups with finite center.

The case of semisimple Lie groups with finite center

One of the most important examples of groups of type $\mathcal{PK}$ with $\mathcal{P}$ amenable and $\mathcal{K}$ compact are non-compact connected semisimple Lie group with finite center. Let $\mathcal{L}$ be such a group. Then $\mathcal{L}$ is unimodular and admits an Iwasawa decomposition $\mathcal{L} = \mathcal{N}\mathcal{A}\mathcal{K}$ with $\mathcal{K}$ compact and $\mathcal{P} = \mathcal{N}\mathcal{A}$ amenable, $\mathcal{K} \cap \mathcal{P} = \{e\}$.

Although $\mathcal{P}$ is not normal in $\mathcal{L}$, we can identify $\mathcal{L}$ with $\mathcal{PK}$ and $\mathcal{P}\backslash\mathcal{L}$ with $\mathcal{K}$ as manifolds, and we have the decomposition

$$
d_{\mathcal{L}}u = d_{\mathcal{P}}x\, d_{\mathcal{K}}s\,,
$$

if $u = xs$, $x \in \mathcal{P}$, $s \in \mathcal{K}$.

As usual, we denote by $\Delta_{\mathcal{P}}$ the modular function of $\mathcal{P}$. This is a $\mathcal{N}$-bi-invariant function on $\mathcal{P}$, and

$$
\Delta_{\mathcal{P}}(x) = e^{-2\rho_{\mathcal{L}}(\log a)}\,, \quad \text{if } x = na \ (n \in \mathcal{N}\,,\ a \in \mathcal{A})\,,
\tag{35}
$$

where $\rho_{\mathcal{L}}$ denotes the half sum of the positive roots weighted by their multiplicities, see, e.g., Helgason [**8**, p 181]) To relate these results with classic theory, we introduce the function ψ_τ, $\tau \in \mathbb{R}$, defined by

$$\psi_\tau(x) = \int_{\mathcal{K}} \Delta_{\mathcal{P}}^{\mathcal{L}}(sx)^{-(\tau+1)/2} ds\,. \tag{36}$$

Denote by $\mathfrak{A}$ the Lie algebra of $\mathcal{A}$. Recall that for each complex valued linear form λ on $\mathfrak{A}$, the elementary spherical function φ_λ is defined by

$$\varphi_\lambda(x) = \int_{\mathcal{K}} \exp\Big(\mathrm{i}\,\lambda\big(\log a(sx)\big) + \rho_{\mathcal{L}}\big(\log a(sx)\big)\Big)\, ds\,,$$

where $a(x)$ denotes the $\mathcal{A}$-component of $x \in \mathcal{L}$ in its $\mathcal{N}\mathcal{A}\mathcal{K}$ decomposition, $\rho_{\mathcal{L}}$ is the half sum of the positive roots, and $\log a$ is the element of $\mathfrak{A}$ whose image by the exponential map is a. See [**8**, Theorem 4.3]. From (35), it follows that

$$\psi_\tau = \varphi_{-\mathrm{i}\,\tau\rho}\,. \tag{37}$$

By [**8**, Lemma 4.4], this implies

$$\psi_\tau(x^{-1}) = \psi_{-\tau}(x). \tag{38}$$

With this notation, Proposition 5.3 leads to the following.

Theorem 5.5. *Let $\mathcal{L}$ be a non-compact connected semisimple Lie group with finite center and Iwasawa decomposition $\mathcal{L} = \mathcal{P}\mathcal{K}$, $\mathcal{P} = \mathcal{N}\mathcal{A}$. Let ϕ be a non-negative function on $\mathcal{L}$ and let K denote the operator of right convolution by ϕ as in* (32)*. Fix $p, q \in [1,\infty]$ with $1/p + 1/q = 1$.*

(i) *Assume that $\phi(us) = \phi(su)$ for all $s \in \mathcal{K}$, $u \in \mathcal{L}$. Then*

$$\sigma_p(K) = \rho_p(K) = \int_{\mathcal{L}} \psi_{1-2/p}(v)\, \phi(v)\, d_{\mathcal{L}} v\,.$$

(ii) *Assume that either $\phi(u) = \phi(su)$ for for all $s \in \mathcal{K}$, $u \in \mathcal{L}$ and*

$$\int_{\mathcal{K}} \left(\int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p}\, \phi(x^{-1}s)\, d_{\mathcal{P}} x \right)^p d_{\mathcal{K}} s < \infty\,,$$

or that $\phi(u) = \phi(us)$ for all $s \in \mathcal{K}$, $u \in \mathcal{L}$ and

$$\int_{\mathcal{K}} \left(\int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p} \phi(sx^{-1})\, d_{\mathcal{P}} x \right)^q d_{\mathcal{K}} s < \infty\,.$$

Then we have

$$\rho_p(K) = \int_{\mathcal{L}} \psi_{1-2/p}(v)\, \phi(v)\, d_{\mathcal{L}} v\,.$$

Proof. Regarding (i), Proposition 5.3(b) gives

$$\sigma_p(K) = \rho_p(K) = \int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/p}\, \phi(v^{-1})\, d_{\mathcal{L}} v\,.$$

Using the fact that ϕ is invariant under $\mathcal{K}$-inner automorphisms and $\Delta_{\mathcal{P}}^{\mathcal{L}}$ right $\mathcal{K}$-invariant, we find that, for any $s \in \mathcal{K}$,

$$\int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/p}\, \phi(v^{-1})\, d_{\mathcal{L}} v = \int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(sv)^{-1/p}\, \phi(v^{-1})\, d_{\mathcal{L}} v\,.$$

Hence, by (37) and (38),

$$\begin{aligned}\sigma_p(K) &= \int_{\mathcal{L}}\int_{\mathcal{K}} \Delta_{\mathcal{P}}^{\mathcal{L}}(sv)^{-1/p}\, d_{\mathcal{K}}s\ \phi(v^{-1})\, d_{\mathcal{L}}v\\ &= \int_{\mathcal{L}} \psi_{-(1-2/p)}(v)\,\phi(v^{-1})\, d_{\mathcal{L}}v\\ &= \int_{\mathcal{L}} \psi_{1-2/p}(v)\,\phi(v)\, d_{\mathcal{L}}v\,.\end{aligned}$$

Regarding (ii), if $\phi(u) = \phi(su)$ for for all $s \in \mathcal{K}\,,\ u \in \mathcal{L}$, then $k_{\mathcal{L}}(u,v) = \phi(v^{-1}u)$ satisfies $k_{\mathcal{L}}(e,vs) = k_{\mathcal{L}}(e,v)$ for all $v \in \mathcal{L}, s \in K$. Hence (34) applies and gives

$$\begin{aligned}\rho_p(K) &= \int_{\mathcal{K}}\int_{\mathcal{P}} \Delta_{\mathcal{P}}(x)^{-1/p}\, k_{\mathcal{L}}(s,x)\, d_{\mathcal{P}}x\, d_{\mathcal{K}}s\\ &= \int_{\mathcal{K}}\int_{\mathcal{P}}\int_{\mathcal{K}} \Delta_{\mathcal{P}}^{\mathcal{L}}(xt)^{-1/p}\, k_{\mathcal{L}}(s,xt)\, d_{\mathcal{P}}x\, d_{\mathcal{K}}t\, d_{\mathcal{K}}s\\ &= \int_{\mathcal{K}}\int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/p}\, k_{\mathcal{L}}(s,v)\, d_{\mathcal{L}}v\, d_{\mathcal{K}}s\\ &= \int_{\mathcal{K}}\int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(v)^{-1/p}\, k_{\mathcal{L}}(e,s^{-1}v)\, d_{\mathcal{L}}v\, d_{K}s\\ &= \int_{\mathcal{K}}\int_{\mathcal{L}} \Delta_{\mathcal{P}}^{\mathcal{L}}(sv)^{-1/p}\, k_{\mathcal{L}}(e,v)\, d_{\mathcal{L}}v\, d_{\mathcal{K}}s\\ &= \int_{\mathcal{L}} \psi_{1-2/p}(v)\,\phi(v)\, d_{\mathcal{L}}v\, d_{\mathcal{K}}s\,.\end{aligned}$$

The last case where $\phi(u) = \phi(us)$ for all $s \in \mathcal{K}\,,\ u \in \mathcal{L}$ is analogous. □

Step 3: General locally compact connected groups. Let us now explain how one can apply Steps 1 and 2 in the case of connected locally compact groups. Any connected locally compact group $\mathcal{G}$ contains a compact normal subgroup $\mathcal{M}$ such that $\mathcal{M}\backslash\mathcal{G}$ is a Lie group. Next, we consider the radical $\mathcal{R}$ of $\mathcal{M}\backslash\mathcal{G}$. By definition, $\mathcal{R}$ is the largest connected closed normal solvable subgroup of $\mathcal{M}\backslash\mathcal{G}$. See, e.g., Paterson [**11**, Prop. 3.7] and Varadarajan [**24**]. Set $\mathcal{S} = \mathcal{R}\backslash[\mathcal{M}\backslash\mathcal{G}]$. Then $\mathcal{S}$ is a connected semisimple Lie group and there exists a central discrete subgroup $\mathcal{Z} \cong \mathbb{Z}^d \subset \mathcal{S}$ (for some integer $d \in \{0,1,\dots\}$) such that $\mathcal{L} = \mathcal{Z}\backslash\mathcal{S}$ is a connected semisimple Lie group with finite center. By this construction we have a surjective group homomorphism from $\mathcal{G}$ to $\mathcal{L}$. The kernel $\mathcal{Q}$ of this homomorphism is a closed normal subgroup, and it is amenable. Indeed, $\mathcal{M}$ is a closed normal subgroup of $\mathcal{Q}$ and $\mathcal{M}\backslash\mathcal{Q}$, viewed as a subgroup of $\mathcal{M}\backslash\mathcal{G}$, contains $\mathcal{R}$ as a closed normal subgroup. Finally, $\mathcal{R}\backslash[\mathcal{M}\backslash\mathcal{Q}] = \mathcal{Z}$. Now, $\mathcal{Z}$ is abelian, $\mathcal{R}$ is solvable and $\mathcal{M}$ is compact. Hence they are all amenable, and so is $\mathcal{Q}$ by [**11**, Prop 1.13]. Of course, we have

$$\mathcal{L} = \mathcal{Q}\backslash\mathcal{G}\,. \tag{39}$$

Let K be a left invariant transition operator on $\mathcal{G}$ as in (30). Let $\mathcal{Q}$ and $\mathcal{L}$ be as above in (39). Proposition 5.1 yields a left invariant transition kernel $k_{\mathcal{L}}$ on the semisimple Lie group $\mathcal{L}$. Now, it is well known that $\mathcal{L}$ is compact if and only if $\mathcal{G}$ is amenable; see e.g. [**11**, Th. 3.8]. Assume that $\mathcal{L}$ is compact. Then we can compute the norm and spectral radius of K on $L^p(\mathcal{G})$ as

$$\begin{aligned}\sigma_p(K) &= \rho_p(K) = \int_{\mathcal{L}} k_{\mathcal{L}}(u,v)\, d_{\mathcal{L}}v \\ &= \int_{\mathcal{L}}\int_{\mathcal{Q}} \Delta_{\mathcal{G}}(x_v^{-1}x_u)^{1/p}\, \Delta_{\mathcal{G}}(y)^{-1/p}\Delta_{\mathcal{G}}(x_v)\, k(x_u, yx_v)\, d_{\mathcal{Q}}y\, d_{\mathcal{L}}v \\ &= \int_{\mathcal{G}} \left(\frac{\Delta_{\mathcal{G}}(x_u)}{\Delta_{\mathcal{G}}(g)}\right)^{1/p} k(x_u,g)\, dg = \int_{\mathcal{G}} \Delta_{\mathcal{G}}(g)^{-1/p}\phi(g^{-1})\, dg.\end{aligned}$$

This yields

$$\sigma_p(K) = \rho_p(K) = \int_{\mathcal{G}} \Delta_{\mathcal{G}}(g)^{-1/p}\phi(g^{-1})\, dg = \int_{\mathcal{G}} \Delta_{\mathcal{G}}(g)^{-1+1/p}\, d\Phi(g)\,.$$

Of course, this conclusion is well known and true in complete generality as soon as $\mathcal{G}$ is amenable. In fact, this formula is at the heart of the proof of Theorem 2.1 in [**15**].

The more interesting case is when $\mathcal{L}$ is *non-compact.* In order to unravel our computation, we need to introduce the following notation. By construction, $\mathcal{L}$ is semisimple with finite center and we let

$$\mathcal{L} = \mathcal{P}\mathcal{K}\,, \quad \mathcal{P} = \mathcal{N}\mathcal{A} \tag{40}$$

be an Iwasawa decomposition as in Step 2 above.

Define $\Delta_{\mathcal{P}}^{\mathcal{G}}$ on G by setting

$$\Delta_{\mathcal{P}}^{G}(g) = \Delta_{\mathcal{P}}^{\mathcal{L}} \circ \pi_{\mathcal{G},\mathcal{L}}(g)\,,$$

where $\Delta_{\mathcal{P}}^{\mathcal{L}}$ is defined by (33) and $\pi_{\mathcal{G},\mathcal{L}}$ denotes the canonical projection from $\mathcal{G}$ onto $\mathcal{L} = \mathcal{Q}\backslash\mathcal{G}$.

Similarly, define

$$\widetilde{\psi}_\tau(g) = \psi_\tau(\pi_{\mathcal{G},\mathcal{L}}(g))$$

with ψ_τ as in (36).

With these definitions, Proposition 5.3 gives the following result.

THEOREM 5.6. *Let $\mathcal{G}$ be a locally compact, connected group and K be a left invariant transition operator on $\mathcal{G}$ as in* (30). *Let ϕ be defined by* (31). *Let $\mathcal{Q}$, $\mathcal{L}$ be as in* (39) *and K be as* (40).

(i) *Assume that $k_{\mathcal{L}}(e, vs) = k_{\mathcal{L}}(e, sv)$ for all $s \in \mathcal{K}$, $v \in \mathcal{L}$. Then*

$$\begin{aligned}\sigma_p(K) = \rho_p(K) &= \int_{\mathcal{G}} \left[\Delta_{\mathcal{P}}^{\mathcal{G}}(g)\Delta_{\mathcal{G}}(g)\right]^{-1/p}\phi(g^{-1})\, d_{\mathcal{G}}g \\ &= \int_{\mathcal{G}} \widetilde{\psi}_{1-2/p}(g)\left[\Delta_{\mathcal{G}}(g)\right]^{-1/p}\phi(g^{-1})\, d_{\mathcal{G}}g.\end{aligned}$$

(ii) *Assume that K is bounded on $L^p(\mathcal{G}, d_{\mathcal{G}}g)$ and $k_{\mathcal{L}}(e, vs) = k_{\mathcal{L}}(e, v)$ for all $s \in \mathcal{K}$, $v \in \mathcal{L}$. Then*

$$\begin{aligned}\rho_p(K) &= \int_{\mathcal{G}} \left[\Delta_{\mathcal{P}}^{\mathcal{G}}(g)\Delta_{\mathcal{G}}(g)\right]^{-1/q}\phi(g)\, d_{\mathcal{G}}g \\ &= \int_{\mathcal{G}} \widetilde{\psi}_{1-2/p}(g)\left[\Delta_{\mathcal{G}}(g)\right]^{-1/q}\phi(g)\, d_{\mathcal{G}}g.\end{aligned}$$

(iii) *Assume that K is bounded on $L^p(\mathcal{G}, d_{\mathcal{G}}g)$ and that $k_{\mathcal{L}}(e, sv) = k_{\mathcal{L}}(e, v)$ for all $s \in \mathcal{K}$, $v \in \mathcal{L}$. Then*

$$\begin{aligned}\rho_p(K) &= \int_{\mathcal{G}} \big[\Delta^{\mathcal{G}}_{\mathcal{P}}(g)\Delta_{\mathcal{G}}(g)\big]^{-1/p} \phi(g^{-1})\, d_{\mathcal{G}}g \\ &= \int_{\mathcal{G}} \widetilde{\psi}_{1-2/p}(g)\big[\Delta_{\mathcal{G}}(g)\big]^{-1/p} \phi(g^{-1})\, d_{\mathcal{G}}g.\end{aligned}$$

Proof. We prove (i). Statements (ii) and (iii) follows by similar arguments. By Propositions 5.1 and 5.3, we have $\sigma_p(K) = \rho_p(K)$, and (letting x_v be, as usual, a representative of $v \in \mathcal{L}$ in $\mathcal{G}$)

$$\begin{aligned}\sigma_p(K) &= \int_{\mathcal{L}} \Delta^{\mathcal{L}}_{\mathcal{P}}(v)^{-1/p}\, k_{\mathcal{L}}(e, v)\, d_{\mathcal{L}}v \\ &= \int_{\mathcal{L}} \int_{\mathcal{Q}} \Delta^{\mathcal{G}}_{\mathcal{P}}(x_v)^{-1/p}\, \Delta_{\mathcal{G}}(yx_v)^{-1/p}\, \Delta_{\mathcal{G}}(x_v)\, k(e, yx_v)\, d_{\mathcal{Q}}y\, d_{\mathcal{L}}v \\ &= \int_{\mathcal{L}} \int_{\mathcal{Q}} \big[\Delta^{\mathcal{G}}_{\mathcal{P}}(yx_v)\Delta_{\mathcal{G}}(yx_v)\big]^{-1/p} \Delta_{\mathcal{G}}(x_v)\, k(e, yx_v)\, d_{\mathcal{Q}}y\, d_{\mathcal{L}}v \\ &= \int_{\mathcal{G}} \big[\Delta^{\mathcal{G}}_{\mathcal{P}}(g)\Delta_{\mathcal{G}}(g)\big]^{-1/p} k(e, g)\, d_{\mathcal{G}}g \\ &= \int_{\mathcal{G}} [\Delta^{\mathcal{G}}_{\mathcal{P}}(g)\Delta_{\mathcal{G}}(g)]^{-1/p} \phi(g^{-1})\, d_{\mathcal{G}}\, g\end{aligned}$$

as desired. The formula using $\widetilde{\psi}_{1-2/p}$ follows by a similar computation using Theorem 5.5. □

5.B. An example: convolutions on $\mathrm{Aff}_+(\mathbb{R}^2)$**.** In this section we mostly follow the notation of Lang [**9**] (except for what [**9**] calls the modular function Δ is Δ^{-1} for us). The group $\mathcal{G} = \mathrm{Aff}_+(\mathbb{R}^2)$ is the group of orientation preserving affine transformations of the plane, that is, $\mathrm{Aff}_+(\mathbb{R}^2) = GL_2^+(\mathbb{R}) \ltimes \mathbb{R}^2$ where $GL_2^+(\mathbb{R})$ is the group of all invertible two by two matrices with real coefficients and positive determinant. The action of $GL_2^+(\mathbb{R})$ on $\mathbb{R}^2$ is the natural action. Thus the product is given by

$$(\mathfrak{m}, \xi)(\mathfrak{m}', \xi') = (\mathfrak{m}\mathfrak{m}', \xi + \mathfrak{m}\xi')\,.$$

Any element $\mathfrak{m}$ of $GL_2^+(\mathbb{R})$ decomposes as $\mathfrak{m} = t \cdot \mathfrak{l}$ where $t > 0$ and t^2 is the determinant of $\mathfrak{m}$, and $\mathfrak{l}$ is in $SL_2(\mathbb{R})$. The group $GL_2^+(\mathbb{R})$ is unimodular with Haar measure $d\mathfrak{m} = t^{-1}\, dt\, d\mathfrak{l}$ where $d\mathfrak{l}$ is a Haar measure on $SL_2(\mathbb{R})$. The group $\mathcal{G} = \mathrm{Aff}_+(\mathbb{R}^2)$ is non-unimodular with left Haar measure

$$d_{\mathcal{G}}g = \det(\mathfrak{m})^{-1}\, d\mathfrak{m}\, d\xi \;\text{ if }\; g = (\mathfrak{m}, \xi)\,,$$

and modular function

$$\Delta_{\mathcal{G}}(g) = \det(\mathfrak{m})^{-1}\,.$$

Observe that we can write

$$\mathcal{G} = \mathrm{Aff}_+(\mathbb{R}^2) = SL_2(\mathbb{R}) \ltimes \big[\mathbb{R} \ltimes \mathbb{R}^2\big]$$

where the action of $\mathbb{R}$ on $\mathbb{R}^2$ is by dilation. When applying the result of Section 5.A to this example, we can take $\mathcal{Q} = \mathbb{R} \ltimes \mathbb{R}^2$ and $\mathcal{L} = SL_2(\mathbb{R})$.

Any element $\mathfrak{l}$ of $\mathcal{L} = SL_2(\mathbb{R})$ has a unique decomposition of the form

$$\mathfrak{l} = \mathfrak{l}_{x,y,s} = \mathfrak{n}_x\, \mathfrak{a}_y\, \mathfrak{k}_s\,,\ x \in \mathbb{R}\,,\ y > 0\,,\ s \in [0, 2\pi)\,,$$

corresponding to the $\mathcal{NAK} = \mathcal{PK}$ Iwasawa decomposition of $SL_2(\mathbb{R})$ where

$$\mathfrak{n}_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\quad \mathfrak{a}_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix},\quad \mathfrak{k}_s = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}.$$

In these coordinates, the Haar measure on $\mathcal{L} = SL_2(\mathbb{R})$ is

$$d_{\mathcal{L}}\mathfrak{l} = y^{-2}\, dx\, dy\, ds\,.$$

and we have

$$\Delta_{\mathcal{P}}^{\mathcal{L}}(\mathfrak{l}) = y^{-1}\,.$$

The pair (x, y) corresponds to a point in the hyperbolic upper half plane $\mathcal{L}/\mathcal{K}$. In particular, the functions on $\mathcal{L}$ which satisfy $\phi_{\mathcal{L}}(\mathfrak{k}\mathfrak{l}) = \phi_{\mathcal{L}}(\mathfrak{l}\mathfrak{k})$ for all $\mathfrak{k} \in \mathcal{K}\,,\ \mathfrak{l} \in \mathcal{L}$ are those of the form

$$\phi_{\mathcal{L}}(\mathfrak{l}_{x,y,s}) = f(z, s) \quad \text{with} \quad z = [x^2 + (y-1)^2]/y \tag{41}$$

whereas the functions on $\mathcal{L}$ which satisfy $\phi_{\mathcal{L}}(\mathfrak{l}) = \phi_{\mathcal{L}}(\mathfrak{l}\mathfrak{k})$ are those of the form

$$\phi_{\mathcal{L}}(\mathfrak{l}_{x,y,s}) = f(x, y). \tag{42}$$

Let $\phi(g)$ be a non-negative function on $\mathcal{G} = \mathrm{Aff}_+(\mathbb{R}^2)$ and consider the operator K on $L^p(\mathcal{G})$ with kernel $k(g, h) = \phi(h^{-1}g)$. Then the corresponding kernel $k_{\mathcal{L}}(u, v) = \phi_{\mathcal{L}}(v^{-1}u)$ on $\mathcal{L} = SL_2(\mathbb{R})$ is given by

$$\phi_{\mathcal{L}}(\mathfrak{l}) = \int_{\mathcal{Q}} t^{2/p}\, \phi\big((t,\xi)^{-1}(l,0)\big)\, \frac{dt\, d\xi}{t^3} = \int_{\mathcal{Q}} t^{2/p} \phi\big((t^{-1}\cdot \mathfrak{l}, t^{-1}\cdot \xi)\big)\, \frac{dt\, d\xi}{t^3}\,.$$

Here, the t in (t, ξ) stands for dilation by the factor t, that is, for t times the identity matrix.

In view of (41) and (42), we are going to look at two classes of functions on $\mathcal{G} = \mathrm{Aff}_+(\mathbb{R}^2)$ in the coordinate system

$$g = g_{t,x,y,s,\xi} = (t \cdot \mathfrak{n}_x\, \mathfrak{a}_y\, \mathfrak{k}_s, \xi)\,,$$

namely, the functions given by

$$\phi(g_{t,x,y,s,\xi}) = f(t, z, s, \xi) \quad \text{with} \quad z = [x^2 + (y-1)^2]/y \tag{43}$$

and those given by

$$\phi(g_{t,x,y,s,\xi}) = f(t, x, y, \xi)\,. \tag{44}$$

Proposition 5.7. *On $\mathcal{G} = \mathrm{Aff}_+(\mathbb{R}^2)$, consider an operator K as in* (30) *with kernel $k(x, y) = \phi(y^{-1}x)$.*

(i) *If ϕ is as in* (43) *then*

$$\sigma_p(K) = \rho_p(K) = \int_{\mathbb{R}^2} \int_{[0,2\pi)} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}_+} f(t, [x^2 + (y^2 - 1)]/y, s, \xi)\, \frac{dt\, dx\, dy\, ds\, d\xi}{t^{1+2/p} y^{2+1/p}}\,.$$

(ii) *If K is bounded on $L^p(\mathcal{G}, d_{\mathcal{G}}g)$ and ϕ is as in* (44) *then*

$$\rho_p(K) = \int_{\mathbb{R}^2} \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}_+} f(t, x, y, \xi)\, \frac{dt\, dx\, dy\, d\xi}{t^{1+2/p} y^{2+1/p}}\,.$$

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Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA
E-mail address: lsc@math.cornell.edu

Institut für Mathematische Strukturtheorie, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
E-mail address: woess@TUGraz.at